Extremal geometry of a Brownian porous medium

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Abstract

The path $W[0,t]$ of a Brownian motion on a d-dimensional torus $\mathbb{T}^d$ run for time t is a random compact subset of $\mathbb{T}^d$. We study the geometric properties of the complement $\mathbb{T}^d \backslash W[0,t]$ as $t \to \infty$ for $d \geq 3$. In particular, we show that the largest region in this complement has linear scale $\varphi_d(t) = [(d \log t)/(d-2)\kappa_d t]^{1/(d-2)}$, where κ_d is the capacity of the unit ball. More specifically, we identify the sets E for which $\mathbb{T}^d \setminus W[0,t]$ contains a translate of $\varphi_d(t)E$, and we count the number of such translates. Furthermore, we derive large deviation principles for the largest inradius as $t \to \infty$ and the ϵ-cover time as $\epsilon \downarrow 0$. Our results, which generalise laws of large numbers proved in [9], are based on a large deviation principle for the shape of the component with largest capacity in $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$, where $W_{\rho(t)}[0,t]$ is the Wiener sausage of radius $\rho(t)$ chosen such that $\varphi_d(t)/(\log t)^{1/d} \ll \rho(t) \ll \varphi_d(t)$ as $t \to \infty$. The idea behind this choice is that $\mathbb{T}^d \backslash W[0,t]$ consists of "lakes" whose linear size is of order $\varphi_d(t)$, connected by narrow "channels" whose linear size is of order at most $\varphi_d(t)/(\log t)^{1/d}$. We also derive large deviation principles for the principal Dirichlet eigenvalue and for the maximal volume of the components of $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$ as $t \to \infty$.

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1 Introduction

1.1 Motivation

Our basic object of study is the complement of a random path:

Question 1. Run a Brownian motion $W = (W(t))_{t\geq 0}$ on a d-dimensional torus $\mathbb{T}^d$, $d \geq 3$. What is the geometry of the random set $\mathbb{T}^d \setminus W[0,t]$ for large t?

Regions with a random boundary have been studied intensively in the literature, and questions such as Question 1 have been approached from a variety of perspectives. Sznitman [20] studies the principal Dirichlet eigenvalue when a Poisson cloud of obstacles is removed from Euclidean space $\mathbb{R}^d$, $d \geq 1$. Van den Berg, Bolthausen and den Hollander [5] consider the large deviation properties of the volume of a Wiener sausage on $\mathbb{R}^d$, $d \geq 2$, and identify the geometric strategies for achieving these large deviations. Probabilistic techniques also play a role in the analysis of deterministic shapes, such as strong circularity in rotor-router and sandpile models shown by Levine and Peres [13], and heat flow in the von Koch snowflake and its relatives analysed by van den Berg and den Hollander [7], van den Berg [3], and van den Berg and Bolthausen [4]. The discrete analogue to Question 1, random walk on a large discrete torus, is connected to the random interlacements model of Sznitman [21].

Question 1 is studied by Dembo, Peres and Rosen [9] for $d \geq 3$ and Dembo, Peres, Rosen and Zeitouni [10] for $d = 2$. In both cases, a law of large numbers is established for the ϵ-cover time (the time for the Brownian motion to come within distance ϵ of every point) as $\epsilon \downarrow 0$. For $d \geq 3$, also the multifractal spectrum of late points is obtained. In the present paper we will consider a large but fixed time t, and we will use a key lemma from [9] to obtain global information about $\mathbb{T}^d \setminus W[0,t]$. Furthermore, throughout the paper we restrict to $d \geq 3$. The behaviour in $d = 2$ is expected to be quite different (see the discussion in Section 1.6.8 below).

A random set is an infinite-dimensional object. Hence issues of measurability may become delicate. In general, events are defined in terms of whether a random closed set intersects a given closed set, or whether a random open set contains a given closed set: see Matheron [14] or Molchanov [16] for a general theory of random sets and questions related to their geometry. On the torus we will parametrize these basic events as

$$\{(x + \varphi E) \cap W[0,t] = \varnothing\} = \left\{x + \varphi E \subset \mathbb{T}^d \setminus W[0,t]\right\}, \qquad x \in \mathbb{T}^d,\ E \subset \mathbb{R}^d \text{ closed} \tag{1.1}$$

(see (1.5) below), where $\varphi > 0$ acts as a scaling factor. The set E in (1.1) plays a role similar to that of a test function, and we will restrict our attention to suitably regular sets E, for instance, compact sets with non-empty interior.

In giving an answer to Question 1, we must distinguish between global properties, such as the size of the largest inradius or the principal Dirichlet eigenvalue of the random set, and local properties, such as whether or not the random set is locally connected. In the present paper we focus on the *global properties* of $\mathbb{T}^d \setminus W[0,t]$. We will therefore be interested in the existence of subsets of $\mathbb{T}^d \setminus W[0,t]$ of a given form:

Question 2. For a given compact set $E \subset \mathbb{R}^d$, what is the probability of the event

$$\big\{\exists x \in \mathbb{T}^d \colon\ x + \varphi E \subset \mathbb{T}^d \setminus W[0,t]\big\} = \bigcup_{x \in \mathbb{T}^d} \big\{x + \varphi E \subset \mathbb{T}^d \setminus W[0,t]\big\} \tag{1.2}$$

formed as the uncountable union of events from (1.1)?

For instance, questions about the inradius can be formulated in terms of Question 2 by setting E to be a ball.

The answer to Question 2 depends on the scaling factor φ. To obtain a non-trivial result we are led to choose $\varphi = \varphi_d(t)$ depending on time, where

$$\varphi_d(t) = \left(\frac{d}{(d-2)\kappa_d}\,\frac{\log t}{t}\right)^{1/(d-2)}, \qquad t > 1, \tag{1.3}$$

and κ_d is the constant

$$\kappa_d = \frac{2\pi^{d/2}}{\Gamma(d/2-1)}. \tag{1.4}$$

We will see that $\varphi_d(t)$ represents the linear size of the largest subsets of $\mathbb{T}^d \setminus W[0,t]$, in the sense that the limiting probability of the event in (1.2) decreases from 1 to 0 as the set E increases from small to large in the sense of small or large capacity (see Section 1.3.3 below).

For a typical point $x \in \mathbb{T}^d$, the event $\big\{x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]\big\}$ in (1.1) is unlikely to occur even when E is fairly small. However, given $E \subset \mathbb{R}^d$, the points $x \in \mathbb{T}^d$ for which $x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]$, i.e., the points that realize the event in (1.2), are atypical, and we can ask whether the subset $x + \varphi_d(t)E$ is likely to form part of a considerably larger subset:

Question 3. Are the points $x \in \mathbb{T}^d$ for which $x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]$ reasonably likely to satisfy $x + \varphi_d(t)E' \subset \mathbb{T}^d \setminus W[0,t]$ for some substantially larger set $E' \supset E$?

Question 3 aims to distinguish between the two qualitative pictures shown in Figure 1, which we call *sparse* and *dense*, respectively. We will show that the answer to Question 3 is no, i.e., the picture is dense as in part (b) of Figure 1.

In a similar spirit, we can ask about *temporal* versus *spatial* avoidance strategies:

Question 4. For a given $x \in \mathbb{T}^d$, does the unlikely event $\big\{x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]\big\}$ arise primarily because the Brownian motion spends an unusually small amount of time near x, or because the Brownian motion spends a typical amount of time near x and simply happens to avoid the set $x + \varphi_d(t)E$?

Questions 3 and 4, though not equivalent, are interrelated: if the Brownian motion spends an unusually small amount of time near x, then it may be plausibly expected to fill the vicinity of x less densely, and vice versa. We will show that the Brownian motion follows a spatial avoidance strategy (the second alternative in Question 4) and that, indeed, the Brownian motion is very likely to spend approximately the same amount of time around all points of $\mathbb{T}^d$.

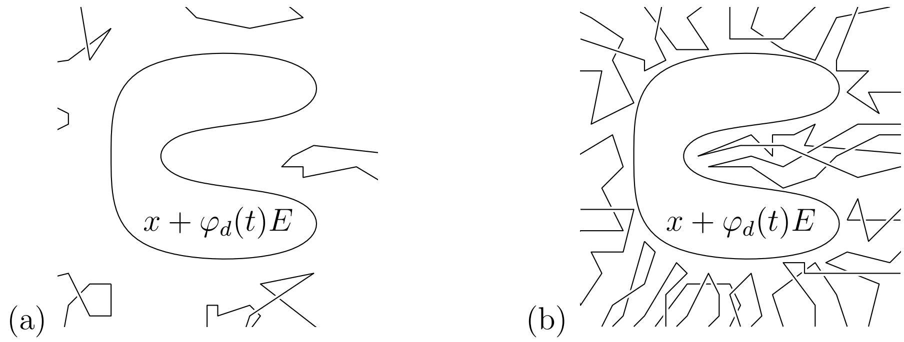

Figure 1: The vicinity of $x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]$. The set in part (a) can be enlarged substantially while remaining a subset of $\mathbb{T}^d \setminus W[0,t]$, the set in part (b) cannot.

The negative answer to Question 3, and the heuristic picture in Figure 1(b), suggest that regions of $\mathbb{T}^d$ where $W[0,t]$ is relatively dense nearly separate the large subsets $x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]$ into disjoint components. Making sense of this heuristic is complicated by the fact that $\mathbb{T}^d \setminus W[0,t]$ is connected almost surely (see Proposition 1.12 below), so that all large subsets belong to the same connected component in $\mathbb{T}^d \setminus W[0,t]$.

Question 5. Can the approximate component structure of the large subsets of $\mathbb{T}^d \setminus W[0,t]$ be captured in a well-defined way?

We will provide a positive answer to Question 5 by enlarging the Brownian path $W[0,t]$ to a Wiener sausage $W_{\rho(t)}[0,t]$ of radius $\rho(t) = o(\varphi_d(t))$. Under suitable hypotheses on the enlargement radius $\rho(t)$ (see (1.16) below) we are able to control certain properties of all the connected components of $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$ simultaneously: for instance, we compute the asymptotics of their maximum possible volume. The well-definedness of this component structure is the fact that (subject to the hypothesis in (1.16)) these properties do not depend on the precise choice of $\rho(t)$.

The existence of a connected component of $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$ having a given property, for instance, having at least a specified volume, involves an uncountable union of the events in (1.2) as E runs over a suitable class of connected sets. Central to our arguments is a discretization procedure that reduces such an uncountable union to a suitably controlled finite union (see Section 3 below).

1.2 Outline

Our main results concern the *extremal geometry* of the set $\mathbb{T}^d \setminus W[0,t]$ as $t \to \infty$. Our key result is a large deviation principle for the *shape of the component with largest capacity* in $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$ as $t \to \infty$, where $W_{\rho(t)}[0,t]$ is the Wiener sausage of radius $\rho(t)$. We also derive a large deviation principle for the maximal volume and principal Dirichlet eigenvalue of the components of $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$, and identify the number of disjoint translates of $\varphi_d(t)E$ in $\mathbb{T}^d \setminus W[0,t]$ for suitable sets E. It turns out that the costs of the various large deviations are *asymmetric*: polynomial in one direction and stretched exponential in the other direction.

Apart from settling the questions raised in Section 1.1, we are interested in the asymptotic behaviour of the following quantities: the largest inradius as $t \to \infty$ and the ϵ-cover time as $\epsilon \downarrow 0$. Laws of large numbers for these quantities were derived in Dembo, Peres and Rosen [9], and we extend these laws of large numbers to large deviation principles.

The remainder of the paper is organised as follows. In Section 1.3 we give definitions and introduce notations. In Sections 1.4 and 1.5 we state our main results: four theorems and six corollaries. In Section 1.6 we discuss these results and state some conjectures. Section 2 contains various estimates on hitting times, hitting numbers and hitting probabilities for Brownian excursions between the boundaries of concentric balls, which serve as key ingredients in the proofs of the main results. Section 3 looks at hitting probabilities of lattice animals, which serve as discrete approximations to continuum sets. The proofs of the main results are given in Sections 4–5. Appendix A contains the proof of two lemmas that are used along the way.

1.3 Definitions and notations

1.3.1 Torus

The d-dimensional *unit torus* $\mathbb{T}^d$ is the quotient space $\mathbb{R}^d/\mathbb{Z}^d$, with the canonical projection map $\pi_0\colon \mathbb{R}^d \to \mathbb{T}^d$. We consider $\mathbb{T}^d$ as a Riemannian manifold in such a way that π_0 is a local isometry. The space $\mathbb{R}^d$ acts on $\mathbb{T}^d$ by translation: given $x = \pi_0(y_0) \in \mathbb{T}^d, y_0, y \in \mathbb{R}^d$, we define $x + y = \pi_0(y_0 + y) \in \mathbb{T}^d$. (Having made this definition, we will no longer need to refer to the projection map π_0, nor to the particular representation of the torus $\mathbb{T}^d$.) Given a set $E \subset \mathbb{R}^d$, a scale factor $\varphi > 0$, and a point $x \in \mathbb{T}^d$ or $x \in \mathbb{R}^d$, we can now define

$$x + \varphi E = \{x + \varphi y\colon\ y \in E\}\,. \tag{1.5}$$

Distances in $\mathbb{T}^d$ and in $\mathbb{R}^d$ are denoted by $d(\cdot,\cdot)$. The distance from a point x to a set E is $d(x,E) = \inf\{d(x,y)\colon\ y \in E\}$. The closed ball of radius r around a point x is denoted by $B(x,r)$, for $x \in \mathbb{T}^d$ or $x \in \mathbb{R}^d$. We will only be concerned with the case $0 < r < \frac{1}{2}$, so that $B(x,r) = x + B(0,r)$ for $x \in \mathbb{T}^d$ and the local isometry $B(0,r) \to B(x,r)$, $y \mapsto x + y$, is one-to-one.

1.3.2 Brownian motion and Wiener sausage

We write $\mathbb{P}_{x_0}$ for the law of the Brownian motion $W = (W(t))_{t\geq 0}$ on $\mathbb{T}^d$ started at $x_0 \in \mathbb{T}^d$, i.e., the Markov process with generator $-\frac{1}{2}\Delta_{\mathbb{T}^d}$, where $\Delta_{\mathbb{T}^d}$ is the Laplace operator for $\mathbb{T}^d$. We can always take $W(t) = x_0 + \tilde{W}(t)$, where $\tilde{W} = (\tilde{W}(t))_{t\geq 0}$ is the standard Brownian motion on $\mathbb{R}^d$ started at 0. For that reason, when $x_0 \in \mathbb{R}^d$ we will also use $\mathbb{P}_{x_0}$ for the law of the Brownian motion on $\mathbb{R}^d$. When the initial point x_0 is irrelevant we will write $\mathbb{P}$ instead of $\mathbb{P}_{x_0}$. The image of the Brownian motion over the time interval $[a,b]$ is denoted by $W[a,b] = \{W(s)\colon\ a \leq s \leq b\}$.

For $r > 0$ and $E \subset \mathbb{R}^d$ or $E \subset \mathbb{T}^d$, we write $E_r = \cup_{x\in E} B(x,r)$ and $E_{-r} = [\cup_{x\in E^c} B(x,r)]^c$. The *Wiener sausage* of radius r run for time t is the r-enlargement $W_r[0,t] = \cup_{s\in[0,t]} B(W(s),r)$.

1.3.3 Capacity

The (Newtonian) *capacity* of a Borel set $E \subset \mathbb{R}^d$, denoted by $\operatorname{Cap} E$, can be defined as

$$\operatorname{Cap} E = \left(\inf_{\mu \in \mathcal{P}(E)} \iint_{E \times E} d\mu(x)\, d\mu(y)\, G(x,y) \right)^{-1}, \tag{1.6}$$

where the infimum runs over the set of probability measures μ on E, and

$$G(x,y) = \frac{\Gamma(d/2-1)}{2\pi^{d/2} d(x,y)^{d-2}} \tag{1.7}$$

is the Green function associated with Brownian motion on $\mathbb{R}^d$ (throughout the paper we restrict to $d \geq 3$). In terms of the constant κ_d from (1.4), we can write $G(x,y) = 1/\kappa_d\, d(x,y)^{d-2}$, and it emerges that $\kappa_d = \operatorname{Cap} B(0,1)$ is the capacity of the unit ball.[1]

The function $E \mapsto \operatorname{Cap} E$ is non-decreasing in E and satisfies the scaling relation

$$\operatorname{Cap}(\varphi E) = \varphi^{d-2} \operatorname{Cap} E, \qquad \varphi > 0, \tag{1.8}$$

and the union bound

$$\operatorname{Cap}(E \cup E') + \operatorname{Cap}(E \cap E') \leq \operatorname{Cap} E + \operatorname{Cap} E'. \tag{1.9}$$

Capacity has an interpretation in terms of Brownian hitting probabilities:

$$\lim_{d(x,0)\to\infty} d(x,0)^{d-2}\, \mathbb{P}_x\big(W[0,\infty) \cap E \neq \varnothing\big) = \frac{\operatorname{Cap} E}{\kappa_d}, \qquad E \subset \mathbb{R}^d \text{ bounded Borel.} \tag{1.10}$$

Thus, capacity measures how likely it is for a set to be hit by a Brownian motion that starts far away. We will make extensive use of estimates similar to (1.10).

If a set E is *polar* – i.e., with probability 1, E is not hit by a Brownian motion started away from E – then $\operatorname{Cap} E = 0$. For instance, any finite or countable union of $(d-2)$-dimensional subspaces has capacity zero.

1.3.4 Sets

The boundary of a set E is denoted by ∂E, the interior by $\operatorname{int}(E)$, and the closure by $\operatorname{clo}(E)$. We write

$$\mathcal{E}_c = \big\{ E \subset \mathbb{R}^d \text{ compact and connected: } \mathbb{R}^d \setminus E \text{ is connected} \big\} \tag{1.11}$$

for those compact sets that can arise as a component of $\mathbb{R}^d \setminus U$ with U a connected open set, and we define

$$\begin{aligned} \mathcal{E}^* = &\big\{ E \subset \mathbb{R}^d \text{ compact: } \operatorname{Cap} E = \operatorname{Cap}(\operatorname{int}(E)) \big\} \\ &\cup \big\{ E \subset \mathbb{R}^d \text{ bounded open: } \operatorname{Cap} E = \operatorname{Cap}(\operatorname{clo}(E)) \big\}. \end{aligned} \tag{1.12}$$

[1] See Port and Stone [18, Section 3.1]. The alternative normalization $\operatorname{Cap} B(0,1) = 1$ is used also, for instance, in Doob [11, Chapter 1.XIII]. This corresponds to replacing $G(x,y)$ by $1/d(x,y)^{d-2}$ in (1.6–1.7).

The condition $\mathrm{Cap}(\mathrm{int}(E)) = \mathrm{Cap}(\mathrm{clo}(E))$ in the definition of $\mathcal{E}^*$ is satisfied when every point of ∂E is a regular point for $\mathrm{int}(E)$, which in turn is satisfied when E satisfies a cone condition at every point (see Port and Stone [18, Chapter 2, Proposition 3.3]). In particular, any finite union of cubes, or any r-enlargement E_r of a compact set, belongs to $\mathcal{E}^*$.

1.3.5 Maximal capacity of a component

A central role will be played by the largest capacity $\kappa^*(t,\rho)$ for a component of $\mathbb{T}^d \setminus W_\rho[0,t]$, defined by

$$\kappa^*(t,\rho) = \sup\left\{\mathrm{Cap}\, E\colon\ x + E \subset \mathbb{T}^d \setminus W_\rho[0,t] \text{ for some } x \in \mathbb{T}^d,\ E \subset \mathbb{R}^d \text{ connected}\right\}. \tag{1.13}$$

Note that by rescaling we have

$$\begin{aligned}&\frac{\kappa^*(t,\rho)}{\varphi_d(t)^{d-2}} \\ &\quad = \sup\left\{\mathrm{Cap}\, E\colon\ x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W_\rho[0,t] \text{ for some } x \in \mathbb{T}^d,\ E \subset \mathbb{R}^d \text{ connected}\right\}.\end{aligned} \tag{1.14}$$

1.4 Component structure

Our first results describe the component structure of our random set. In formulating these results we will use the abbreviation (see Figure 2(a))

$$J_d(\kappa) = \frac{d}{d-2}\left(1 - \frac{\kappa}{\kappa_d}\right), \qquad \kappa \geq 0. \tag{1.15}$$

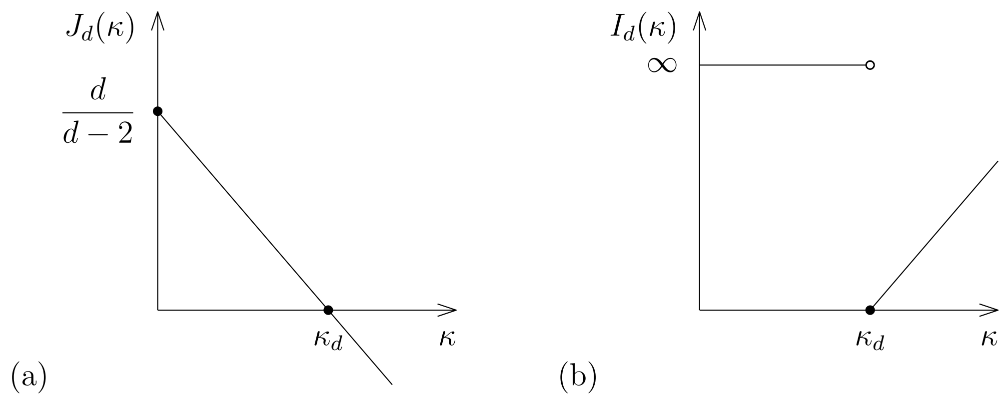

Figure 2: (a) The function $\kappa \mapsto J_d(\kappa)$ in (1.15). (b) The rate function $\kappa \mapsto I_d(\kappa)$ in (1.17).

Our first theorem quantifies the likelihood of finding sets of large capacity that are not hit by the Wiener sausage.

Theorem 1.1. *Fix a positive function $t \mapsto \rho(t)$ satisfying*

$$\lim_{t\to\infty} \frac{\rho(t)}{\varphi_d(t)} = 0, \qquad \lim_{t\to\infty} \frac{(\log t)^{1/d}\rho(t)}{\varphi_d(t)} = \infty. \tag{1.16}$$

Then the family $\mathbb{P}(\kappa^*(t,\rho)/\varphi_d(t)^{d-2} \in \cdot)$, $t > 1$, *satisfies the LDP on* $[0,\infty]$ *with rate* $\log t$ *and rate function (see Figure 2(b))*

$$I_d(\kappa) = \begin{cases} -J_d(\kappa), & \kappa \geq \kappa_d, \\ \infty, & \kappa < \kappa_d, \end{cases} \tag{1.17}$$

with the convention that $I_d(\infty) = \infty$.

The counterpart of Theorem 1.1 for small capacities is contained in the following two theorems, which show that components of small capacity are very likely to exist and to be numerous. Let $\chi_\rho(t,\kappa)$ denote the number of components C of $\mathbb{T}^d \setminus W_\rho[0,t]$ such that C contains some ball of radius ρ and has the form $C = x + \varphi_d(t)E$ for a connected open set E with $\operatorname{Cap} E \geq \kappa$.

Theorem 1.2. *Fix a positive function* $t \mapsto \rho(t)$ *satisfying* (1.16), *and let* $\kappa < \kappa_d$. *Then*

$$\lim_{t\to\infty} \frac{\log \chi_{\rho(t)}(t,\kappa)}{\log t} = J_d(\kappa) \qquad \textit{in } \mathbb{P}\textit{-probability.} \tag{1.18}$$

Theorem 1.3. *Fix positive functions* $t \mapsto \rho(t)$ *and* $t \mapsto h(t)$ *satisfying*

$$\lim_{t\to\infty} \frac{\rho(t)}{\varphi_d(t)} = 0, \qquad \lim_{t\to\infty} \frac{\log[h(t)/\varphi_d(t)]}{\log t} = 0, \tag{1.19}$$

and collections of points $(S(t))_{t>1}$ *in* $\mathbb{T}^d$ *such that* $\max_{x\in\mathbb{T}^d} d(x,S(t)) \leq h(t)$ *for all* $t > 1$. *Then, for any* $E \subset \mathbb{R}^d$ *compact with* $\operatorname{Cap} E < \kappa_d$,

$$\log \mathbb{P}\left((x+\varphi_d(t)E) \cap W_{\rho(t)}[0,t] \neq \varnothing \ \forall x \in S(t)\right) \leq -t^{J_d(\operatorname{Cap} E)+o(1)}, \qquad t \to \infty. \tag{1.20}$$

The next two theorems identify the shapes of components of $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$. For $E \subset E'$ a pair of nested compact connected subsets of $\mathbb{R}^d$ we say that a component C of $\mathbb{T}^d \setminus W_\rho[0,t]$ satisfies condition $(\mathcal{C}(t,\rho,E,E'))$ when

$$C = x + \varphi_d(t)U \qquad \text{where } x \in \mathbb{T}^d \text{ and } E \subset U \subset E'. \tag{$\mathcal{C}(t,\rho,E,E')$}$$

Define $\chi_\rho(t,E,E')$ to be the number of components of $\mathbb{T}^d \setminus W_\rho[0,t]$ satisfying condition $(\mathcal{C}(t,\rho,E,E'))$, and define $F_\rho(t,E,E')$ to be the event

$$F_\rho(t,E,E') = \left\{ \begin{array}{l} \text{There exists a component } C = x + \varphi_d(t)U \text{ of } \mathbb{T}^d \setminus W_\rho[0,t] \\ \text{satisfying condition } (\mathcal{C}(t,\rho,E,E')), \text{ and any other com-} \\ \text{ponent } C' = x' + \varphi_d(t)U' \text{ has } \operatorname{Cap} U' < \operatorname{Cap} U. \end{array} \right\} \tag{1.21}$$

In words, $F_\rho(t,E,E')$ is the event that $\mathbb{T}^d \setminus W_\rho[0,t]$ contains a component sandwiched between $x + \varphi_d(t)E$ and $x + \varphi_d(t)E'$, and any other component has smaller capacity (when viewed as a subset of $\mathbb{R}^d$).

Theorem 1.4. *Fix a positive function $t \mapsto \rho(t)$ satisfying* (1.16)*, let $E \in \mathcal{E}_c$, and let $\delta > 0$. If* $\operatorname{Cap} E \geq \kappa_d$*, then*

$$\lim_{t\to\infty} \frac{\log \mathbb{P}\big(F_{\rho(t)}(t, E, E_\delta)\big)}{\log t} = J_d(\operatorname{Cap} E) = -I_d(\operatorname{Cap} E), \tag{1.22}$$

while if $\operatorname{Cap} E < \kappa_d$*, then*

$$\lim_{t\to\infty} \frac{\log \chi_{\rho(t)}(t, E, E_\delta)}{\log t} = J_d(\operatorname{Cap} E) \qquad \textit{in } \mathbb{P}\textit{-probability.} \tag{1.23}$$

The following corollary is the special case of Theorem 1.3 with $S(t) = \mathbb{T}^d$ and $\rho(t) = 0$.

Corollary 1.5. *Let $E \subset \mathbb{R}^d$ be compact with* $\operatorname{Cap} E < \kappa_d$*. Then*

$$\log \mathbb{P}\left(\nexists\, x \in \mathbb{T}^d\colon\ x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]\right) \leq -t^{J_d(\operatorname{Cap} E)+o(1)}, \qquad t \to \infty. \tag{1.24}$$

Theorems 1.1–1.4 allow us to answer Question 2, subject to a regularity condition on the set E. For $E \subset \mathbb{R}^d$, let $\chi(t, E)$ denote the maximal number of disjoint translates $x + \varphi_d(t)E$ in $\mathbb{T}^d \setminus W[0,t]$.

Corollary 1.6. *Suppose that $E \in \mathcal{E}^*$. Then*

$$\lim_{t\to\infty} \mathbb{P}\left(\exists\, x \in \mathbb{T}^d\colon\ x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]\right) = \begin{cases} 1, & \operatorname{Cap} E < \kappa_d, \\ 0, & \operatorname{Cap} E > \kappa_d. \end{cases} \tag{1.25}$$

Furthermore,

$$\lim_{t\to\infty} \frac{\log \mathbb{P}(\exists\, x \in \mathbb{T}^d\colon\ x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t])}{\log t} = J_d(\operatorname{Cap} E) \vee 0, \tag{1.26}$$

and if $\operatorname{Cap} E < \kappa_d$*, then*

$$\lim_{t\to\infty} \frac{\log \chi(t, E)}{\log t} = J_d(\operatorname{Cap} E) \quad \textit{in } \mathbb{P}\textit{-probability.} \tag{1.27}$$

1.5 Geometric structure

Having described the components in terms of their capacities in Section 1.4, we are ready to look at the geometric structure of our random set. Our first corollary concerns the maximal volume of a component of $\mathbb{T}^d \setminus W_\rho[0,t]$, which we denote by $V(t,\rho)$. Volume is taken w.r.t. the d-dimensional Lebesgue measure, and we write $V_d = \operatorname{Vol} B(0,1)$ for the volume of the d-dimensional unit ball.

Corollary 1.7. *Subject to* (1.16)*, the family $\mathbb{P}(V(t,\rho(t))/\varphi_d(t)^d \in \cdot)$, $t > 1$, satisfies the LDP on $(0,\infty)$ with rate $\log t$ and rate function*

$$I_d^{\text{volume}}(v) = I_d\left(\kappa_d (v/V_d)^{(d-2)/d}\right). \tag{1.28}$$

Moreover, for $v < V_d$,

$$\log \mathbb{P}\big(V(t,\rho(t))/\varphi_d(t)^d < v\big) \leq -t^{J_d(\kappa_d (v/V_d)^{(d-2)/d})+o(1)}, \qquad t \to \infty. \tag{1.29}$$

Our second corollary concerns $\lambda(t,\rho) = \lambda(\mathbb{T}^d \setminus W_\rho[0,t])$, the principal Dirichlet eigenvalue of $\mathbb{T}^d \setminus W_\rho[0,t]$, where by $\lambda(E)$ (for $E \subset \mathbb{T}^d$ or $E \subset \mathbb{R}^d$) we mean the principal eigenvalue of the operator $-\frac{1}{2}\Delta_E$ with Dirichlet boundary conditions on ∂E. We write $\lambda_d = \lambda(B(0,1))$ for the principal Dirichlet eigenvalue of the d-dimensional unit ball.

Corollary 1.8. *Subject to* (1.16), *the family* $\mathbb{P}(\varphi_d(t)^2\lambda(t,\rho(t)) \in \cdot)$, $t>1$, *satisfies the LDP on* $(0,\infty)$ *with rate* $\log t$ *and rate function*

$$I_d^{\text{Dirichlet}}(\lambda) = I_d\left(\kappa_d(\lambda_d/\lambda)^{(d-2)/2}\right). \tag{1.30}$$

Moreover, for $\lambda > \lambda_d$,

$$\log \mathbb{P}\big(\varphi_d(t)^2\lambda(t,\rho(t)) \geq \lambda\big) \leq -t^{J_d(\kappa_d(\lambda_d/\lambda)^{(d-2)/2})+o(1)}, \qquad t \to \infty. \tag{1.31}$$

Our last two corollaries concern the largest inradius of $\mathbb{T}^d \setminus W[0,t]$,

$$\rho_{\text{in}}(t) = \sup_{x \in \mathbb{T}^d} d(x, W[0,t]) = \sup\left\{\rho \geq 0\colon\ \mathbb{T}^d \setminus W_\rho[0,t] \neq \varnothing\right\}, \tag{1.32}$$

and the ϵ-cover time,

$$\mathcal{C}_\epsilon = \sup_{x \in \mathbb{T}^d} \inf\left\{t \geq 0\colon\ d(x, W[0,t]) \leq \epsilon\right\} = \inf\left\{t \geq 0\colon\ \rho_{\text{in}}(t) \leq \epsilon\right\}. \tag{1.33}$$

For the latter we need the scaling function

$$\psi_d(\epsilon) = \frac{\epsilon^{-(d-2)}\log(1/\epsilon)}{\kappa_d}. \tag{1.34}$$

Corollary 1.9. *The family* $\mathbb{P}(\rho_{\text{in}}(t)/\varphi_d(t) \in \cdot)$, $t>1$, *satisfies the LDP on* $(0,\infty)$ *with rate* $\log t$ *and rate function*

$$I_d^{\text{inradius}}(r) = I_d(\kappa_d\, r^{d-2}). \tag{1.35}$$

Moreover, for $0 < r < 1$,

$$\log \mathbb{P}\big(\rho_{\text{in}}(t)/\varphi_d(t) < r\big) \leq -t^{J_d(\kappa_d\, r^{d-2})+o(1)}, \qquad t \to \infty. \tag{1.36}$$

Corollary 1.10. *The family* $\mathbb{P}(\mathcal{C}_\epsilon/\psi_d(\epsilon) \in \cdot\,)$, $0<\epsilon<1$, *satisfies the LDP on* $(0,\infty)$ *with rate* $\log(1/\epsilon)$ *and rate function*

$$I_d^{\text{cover}}(u) = \begin{cases} u-d, & u \geq d, \\ \infty, & 0 < u < d. \end{cases} \tag{1.37}$$

Moreover, for $0 < u < d$,

$$\log \mathbb{P}\big(\mathcal{C}_\epsilon/\psi_d(\epsilon) < u\big) \leq -\epsilon^{-(d-u)+o(1)}, \qquad \epsilon \downarrow 0. \tag{1.38}$$

Corollary 1.10 is equivalent to Corollary 1.9 because of the relation $\{\rho_{\text{in}}(t) > \epsilon\} = \{\mathcal{C}_\epsilon > t\}$ and the asymptotics

$$\varphi_d(u\psi_d(\epsilon)) \sim \left(\frac{u}{d}\right)^{1/(d-2)}\epsilon, \qquad \psi_d(r\varphi_d(t)) \sim \frac{t}{dr^{d-2}}, \qquad \epsilon \downarrow 0,\ t \to \infty,\ u,r > 0. \tag{1.39}$$

1.6 Discussion

1.6.1 Upward versus downward deviations and the role of $J_d(\kappa)$

Theorem 1.1 says that the region with largest capacity not hit by the Wiener sausage of radius $\rho(t)$ lives on scale $\varphi_d(t)$, and that upward large deviations on this scale have a cost that decays *polynomially* in t. Theorem 1.2 identifies how many components of small capacity exist. This number grows *polynomially* in t.

According to Corollary 1.5, the number $\chi(t, E)$ of scaled translates $x + \varphi_d(t)E$ is extremely unlikely to be zero: the cost is *stretched exponential* in t. Theorem 1.3 strengthens this result and thereby indicates its essential ingredients: the bound remains valid even when the Brownian motion is enlarged to a Wiener sausage of radius $\rho(t)$ and the points x are restricted to lie on grid of spacing $h(t)$. The second part of condition (1.19) says that $h(t)$ can be as large as $\varphi_d(t)t^{o(1)}(\ll 1)$, i.e., the number of translates can be as small as $1/[\varphi_d(t)t^{o(1)}]^d(\gg 1)$. For smaller grids, i.e., for larger numbers of translates, the event is even more unlikely.

Theorems 1.1–1.3 are linked by the heuristic that components of the form $x+\varphi_d(t)E$ appear according to a Poisson point process with total intensity $t^{J_d(\operatorname{Cap} E)+o(1)}$. When $\operatorname{Cap} E > \kappa_d$ we have $J_d(\operatorname{Cap} E) < 0$, and the likelihood of even a single such component is $t^{-|J_d(\operatorname{Cap} E)|+o(1)}$, as in Corollary 1.6. When $\operatorname{Cap} E < \kappa_d$ we have $J_d(\operatorname{Cap} E) > 0$, and a Poisson random variable X of mean $t^{J_d(\operatorname{Cap} E)+o(1)}$ will satisfy $X = t^{J_d(\operatorname{Cap} E)+o(1)}$ with high probability and $\mathbb{P}(X = 0) = \exp\left[-t^{J_d(\operatorname{Cap} E)+o(1)}\right]$, as in Theorems 1.2–1.3. Based on this heuristic, we conjecture that the inequality in (1.20), and the corresponding inequalities in (1.29), (1.31), (1.36) and (1.38), are all equalities asymptotically.

Theorem 1.4 completes the picture from Theorems 1.1–1.3 by showing that components can approximate any shape $E \in \mathcal{E}_c$.

1.6.2 Components and the role of $\rho(t)$

Theorems 1.1–1.4 concern components of the form $x + \varphi_d(t)E$. We begin by remarking that, with high probability, all components have this form:

Proposition 1.11. *Assume* (1.16). *Let* $\operatorname{Wrap}(t, \rho)$ *be the event that* $\mathbb{T}^d \setminus W_\rho[0, t]$ *has a component* C *that, when considered as a Riemannian manifold with its intrinsic metric, is not the isometric image* $x + E$ *of a bounded subset* E *of* $\mathbb{R}^d$*. Then*

$$\lim_{t\to\infty} \frac{\log \mathbb{P}\left(\operatorname{Wrap}(t, \rho(t))\right)}{\log t} = -\infty. \tag{1.40}$$

Informally, such a component must "wrap around" the torus, so that the local isometry from $\mathbb{R}^d$ to $\mathbb{T}^d$ is not a global isometry. Proposition 1.11 means that, apart from a negligible event, we may sensibly consider the components as subsets of $\mathbb{R}^d$ and discuss their capacities as defined in (1.6).

Collectively, Theorems 1.1–1.4, Corollaries 1.7–1.8 and Proposition 1.11 show that $\mathbb{T}^d \setminus W_{\rho(t)}[0, t]$ has a *component structure*, with well-defined bounds on the capacities, volumes and principal Dirichlet eigenvalues of these components. These results provide an answer to Question 5.

By contrast, the choice $\rho(t) = 0$ does not give a component structure at all:

Proposition 1.12. *With probability* 1*, the set* $\mathbb{T}^d \setminus W[0,t]$ *is path-connected, open and dense for every* t*, and the set* $\mathbb{T}^d \setminus W\,[0,\infty)$ *is path-connected, locally path-connected and dense.*

The picture linking Propositions 1.11–1.12 is that the set $\mathbb{T}^d \setminus W[0,t]$ consists of "lakes" whose linear size is of order $\varphi_d(t)$, connected by narrow "channels" whose linear size is at most $\varphi_d(t)/(\log t)^{1/d}$. By inflating the Brownian motion to a Wiener sausage of radius $\rho(t)$ with $\varphi_d(t)/(\log t)^{1/d} \ll \rho(t) \ll \varphi_d(t)$, we effectively block off these channels, so that $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$ consists of disjoint lakes.

Proposition 1.12 shows that some lower bound on $\rho(t)$ is necessary for the results of Theorems 1.1–1.4, Corollaries 1.7–1.8 and Proposition 1.11 to hold.[2] It would be of interest to know whether the condition $\rho(t) \gg \varphi_d(t)/(\log t)^{1/d}$ can be relaxed, i.e., whether the true size of the channels is of smaller order than $\varphi_d(t)/(\log t)^{1/d}$. By analogy with the random interlacements model (see Section 1.6.3 below), the relevant regime to study would be $\varphi_d(t)/(\log t)^{1/(d-2)} \ll \rho(t) \ll \varphi_d(t)/(\log t)^{1/d}$.

1.6.3 A comparison with random interlacements

The discrete analogue of $\mathbb{T}^d \setminus W[0,t]$ is the complement $\mathbb{T}_N^d \setminus S[0,n]$ of the path of a random walk $S = (S(n))_{n\in\mathbb{N}_0}$ on a large discrete torus $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$. The spatial scale being fixed by discretization, it is necessary to take $N \to \infty$ and $n \to \infty$ simultaneously, and the choice $n = uN^d$ for $u \in (0,\infty)$ has been extensively studied: see for instance Benjamini and Sznitman [2], Sznitman [21] and Sidoravicius and Sznitman [19]. Teixeira and Windisch [22] prove that the law of $\mathbb{T}_N^d \setminus S[0,uN^d]$, seen *locally* from a *typical* point, converges in law as $N \to \infty$: with X drawn uniformly from $\mathbb{T}_N^d$,

$$\lim_{N\to\infty} \mathbb{P}_{x_0}\left(X + E \subset \mathbb{T}_N^d \setminus S[0,uN^d]\right) = e^{-u\,\mathrm{Cap}_{\mathbb{Z}^d} E}, \qquad E \subset \mathbb{Z}^d \text{ finite}, \tag{1.41}$$

where $\mathrm{Cap}_{\mathbb{Z}^d} E$ is the discrete capacity. The right-hand side of (1.41) is the hitting probability

$$\mathbb{P}(E \subset \mathbb{Z}^d \setminus \mathcal{I}^u = \varnothing) = e^{-u\,\mathrm{Cap}_{\mathbb{Z}^d} E} \tag{1.42}$$

for the *random interlacements* model with parameter u introduced by Sznitman [21]. Here, the set $\mathcal{I}^u \subset \mathbb{Z}^d$ is the union of a certain Poisson point process of random walk paths, with an intensity proportional to the parameter u. The random interlacements model has a critical value $u_* \in (0,\infty)$ such that $\mathbb{Z}^d \setminus \mathcal{I}^u$ has an unbounded component a.s. when $u < u_*$ and has only bounded components a.s. when $u > u_*$.

The continuous analogue of (1.41) is the probability of the event in (1.1) with the scaling factor $\varphi = t^{-1/(d-2)}$ instead of $\varphi = \varphi_d(t)$. Our methods (see Propositions 2.1 and 2.4 below) yield

$$\lim_{t\to\infty} \mathbb{P}_{x_0}\left(X + t^{-1/(d-2)}E \subset \mathbb{T}^d \setminus W[0,t]\right) = e^{-\mathrm{Cap}\,E}, \qquad E \subset \mathbb{R}^d \text{ compact} \tag{1.43}$$

[2]The choice $\rho(t) = 0$ makes the eigenvalue result in Corollary 1.8 false for $d \geq 4$, since Brownian motion is a polar set for $d \geq 4$. However, for $d = 3$ the eigenvalue $\lambda(t,0)$ is non-trivial even when $\rho(t) = 0$, and we conjecture that Corollary 1.8 remains valid, i.e., the eigenvalue is determined primarily by the large unhit lakes in $\mathbb{T}^d \setminus W[0,t]$, and not by the narrow channels connecting them. See the rough estimates in van den Berg, Bolthausen and den Hollander [6].

for X drawn uniformly from $\mathbb{T}^d$, which implies that the random set $\mathbb{T}^d \setminus W[0,t]$, seen locally from a typical point, converges in law (see Molchanov [16, Theorem 6.5] for a discussion of convergence in law for random sets). As with the random interlacements $\mathcal{I}^u$, the limiting random set for (1.43) can be constructed from a Poisson point process of Brownian motion paths.

Because of scale invariance, no parameter is needed in (1.43). Indeed, the continuous model corresponds to a rescaled limit of the discrete model when N and u are replaced by kN and u/k^{d-2}, respectively, with $k \to \infty$. In this rescaling the parameter u tends to zero. Thus, $\mathbb{Z}^d \setminus \mathcal{I}^u$ must lose its finite component strucure, which is in accordance with the connectedness result Proposition 1.12.

Expanding to a Wiener sausage can be interpreted as reintroducing a kind of discretization. However, because of (1.16), the spatial scale $\rho(t)$ of this discretization is much larger than the spatial scale $t^{-1/(d-2)}$ corresponding to (1.43) (cf. Section 1.6.2).

1.6.4 Corollaries of the capacity bounds

Corollary 1.6 summarizes for which set E a subset $x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]$ can be expected to exist: according to Theorems 1.1–1.4, subsets of large capacity are unlikely to exist, whereas subsets of small capacity are numerous. In particular, Corollary 1.6 answers Question 2 subject to the regularity condition $E \in \mathcal{E}^*$.

Corollaries 1.7–1.8 follow from Theorems 1.1–1.4 with the help of the isoperimetric inequalities

$$\frac{\operatorname{Cap} E}{\kappa_d} \geq \left(\frac{\operatorname{Vol} E}{V_d}\right)^{(d-2)/d} \geq \left(\frac{\lambda_d}{\lambda(E)}\right)^{(d-2)/2}, \qquad E \subset \mathbb{R}^d \text{ bounded open}, \tag{1.44}$$

where we recall that κ_d, V_d, λ_d are the capacity, volume and principal Dirichlet eigenvalue of $B(0,1)$. The first inequality is the Poincaré-Faber-Szegö inequality, which says that among all sets with a given volume the ball has the smallest capacity. The second inequality is the Faber-Krahn theorem, which says that among all sets of a given volume the ball has the smallest Dirichlet eigenvalue.[3] Comparing with Theorem 1.1, the most efficient way to produce a component having a given large volume (or small principal Dirichlet eigenvalue) is for that component to be a ball.

Equality holds throughout (1.44) when E is a ball, and the lower bounds in Corollaries 1.7–1.8, together with Corollaries 1.9–1.10, follow by specializing Theorems 1.1–1.4 to that case.

The large deviation principles in Theorem 1.1 and Corollaries 1.7–1.10 each imply a weak law of large numbers, e.g. $\lim_{t\to\infty} \kappa^*(t, \rho(t))/\varphi_d(t)^{d-2} = 1$ in $\mathbb{P}$-probability. The weak laws of large numbers implied by Corollaries 1.9–1.10 were proved in Dembo, Peres and Rosen [9] in the stronger form $\lim_{t\to\infty} \rho_{\text{in}}(t)/\varphi_d(t) = 1$ and $\lim_{t\to\infty} \mathcal{C}_\epsilon/\psi_d(\epsilon) = d$ $\mathbb{P}$-almost surely. The L^1-version of this convergence is proved in van den Berg, Bolthausen and den Hollander [6]. Note that none of these forms are equivalent: for instance, almost sure convergence does not follow from Corollaries 1.9–1.10, since the sum $\sum_{t\in\mathbb{N}} \exp[-I_d(\kappa)\log t]$ fails to converge when $I_d(\kappa)$ is small.

[3] See e.g. Bandle [1, Theorems II.2.3 and III.3.8] or Pólya and Szegö [17, Section I.1.12]. These references consider the capacity only when $d = 3$, but their methods apply for all $d \geq 3$.

1.6.5 The maximal diameter of a component

There is no analogue of Corollary 1.7 for the maximal diameter instead of the maximal volume. The capacity and the diameter are related by $\operatorname{Cap} E \leq \kappa_d (\operatorname{diam} E)^{d-2}$. However, there is no inequality in the reverse direction: a set of fixed capacity can have an arbitrarily large diameter. It turns out that the maximal diameter of the components of $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$ is of larger order than $\varphi_d(t)$. More precisely, suppose that $\rho(t) = o(\varphi_d(t))$, and let $D(t, \rho(t))$ denote the largest diameter of a component of $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$. Then $\lim_{t\to\infty} D(t,\rho(t))/\varphi_d(t) = \infty$ in $\mathbb{P}$-probability. Indeed, choose a compact connected set E of zero capacity and large diameter, say $E = [0, L] \times \{0\}^{d-1}$ with L large. Then, by Theorem 1.3, $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$ has a component containing $x + \varphi_d(t)E$, for some x, with a very high probability.

Furthermore, we conjecture that the "actual" asymptotic size of $D(t, \rho(t))$ (which, as we have just shown, must be of larger order than $\varphi_d(t)$) depends on the choice of $\rho(t)$, and is therefore *ill-defined* in the sense of Question 5.

1.6.6 The second-largest component

The component of second-largest capacity (or second-largest volume, principal Dirichlet eigenvalue, or inradius) has a different large deviation behaviour, due to the fact that $E \mapsto \operatorname{Cap} E$ is not additive. Indeed, typically $\operatorname{Cap}(E^{(1)} \cup E^{(2)}) < \operatorname{Cap}(E^{(1)}) + \operatorname{Cap}(E^{(2)})$, even for disjoint sets $E^{(1)}, E^{(2)}$. In the case of concentric spheres, $\operatorname{Cap}(\partial B(0,r_1) \cup \partial B(0,r_2)) = \max\{\operatorname{Cap}(\partial B(0,r_1)), \operatorname{Cap}(\partial B(0,r_2))\}$. It follows that the most efficient way to produce two large but disjoint components is to have them almost touching.

1.6.7 Answers to Questions 1–5

The results in this paper give a partial answer to Question 1.

Question 2 is answered by Corollary 1.6 subject to $E \in \mathcal{E}^*$, $\operatorname{Cap} E \neq \kappa_d$ (see also Section 3 for results that are simultaneous over a certain class of sets E).

The resolution to Question 3, namely, the fact that the dense picture in Figure 1(b) applies, is provided by Corollary 1.6. If $E \subset E'$ with $\operatorname{Cap} E' \geq \operatorname{Cap} E + \delta$, $\delta > 0$, and $E, E' \in \mathcal{E}^*$, then, compared to subsets of the form $x + \varphi_d(t)E$, subsets of the form $x + \varphi_d(t)E'$ are much less numerous (when $\operatorname{Cap} E < \kappa_d$) or much less probable (when $\operatorname{Cap} E \geq \kappa_d$). Moreover, if (1.16) holds, then Theorems 1.1–1.2 answer Question 3 *simultaneously* over all possible sets E'.

The answer to Question 4, namely, that the Brownian motion follows a spatial avoidance strategy, will follow from Proposition 2.1 below.

Finally (with the caveat discussed in Section 1.6.5), Theorems 1.1–1.4, Corollaries 1.7–1.8 and Proposition 1.11 provide the answer to Question 5.

1.6.8 Two dimensions

It remains a challenge to extend the results in this paper to $d = 2$. In contrast to $d \geq 3$, the large subsets of $\mathbb{T}^2 \setminus W[0,t]$ are expected to arise because of a *temporal* avoidance strategy and to resemble the *sparse* picture of Figure 1(a) (see Questions 3–4). A law of

large numbers for the cover time is derived in Dembo, Peres, Rosen and Zeitouni [10]. By the relation $\{\rho_{\mathrm{in}}(t) > \epsilon\} = \{\mathcal{C}_\epsilon > t\}$, this implies a law of large numbers for a certain rescaled function of the inradius. However, differences in scaling properties mean that this law of large numbers gives much less precise geometric information. Rough asymptotics for the average principal Dirichlet eigenvalue are derived in van den Berg, Bolthausen and den Hollander [6].

2 Brownian excursions

In this section we list a few properties of Brownian excursions that will be needed as we go along. Section 2.1 looks at the times and the numbers of excursions between the boundaries of two concentric balls, Section 2.2 estimates the hitting probabilities of these excursions in terms of capacity, while Section 2.3 collects a few elementary properties of capacity.

2.1 Counting excursions between balls

Excursion times. Let $x \in \mathbb{T}^d$ and $0 < r < R < \frac{1}{2}$. Regard these values as fixed for the moment. Set $T_0 = \inf\{t \geq 0\colon\ W(t) \in \partial B(x,R)\}$ and, for $i \in \mathbb{N}$, define recursively the hitting times (see Figure 3)

$$\begin{aligned} T_i' &= \inf\{t \geq T_{i-1}\colon\ W(t) \in \partial B(x,r)\}\,, \\ T_i &= \inf\{t \geq T_i'\colon\ W(t) \in \partial B(x,R)\}\,. \end{aligned} \tag{2.1}$$

We call $W[T_i', T_i]$ the i^{th} *excursion from* $\partial B(x,r)$ *to* $\partial B(x,R)$, and write $\xi_i'(x) = W(T_i')$, $\xi_i(x) = W(T_i)$ for its starting and ending points.[4]

Set

$$\begin{aligned} &\tau_0(x,r,R) = \tau_0'(x,r,R) = T_0(x), \\ &\tau_i(x,r,R) = T_i - T_{i-1},\ \tau_i'(x,r,R) = T_i' - T_{i-1}, \quad i \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Thus, $\tau_i(x,r,R)$ is the duration of the i^{th} excursion from $\partial B(x,R)$ to itself via $\partial B(x,r)$, while $\tau_i'(x,r,R) < \tau_i(x,r,R)$ is the duration of the i^{th} excursion from $\partial B(x,R)$ to $\partial B(x,r)$.
(All the variables $T_i, T_i', \xi_i, \xi_i', \tau_i, \tau_i'$ depend on all the parameters x, r, R. Nevertheless, in our notation we only indicate some of these dependencies.)

Excursion numbers. Define

$$N(x,t,r,R) = \max\{i \in \mathbb{N}_0\colon\ T_i \leq t\} = \max\left\{j \in \mathbb{N}_0\colon\ \sum_{i=0}^{j} \tau_i(x,r,R) \leq t\right\}, \tag{2.3}$$

$$N'(x,t,r,R) = \max\left\{j \in \mathbb{N}_0\colon\ \sum_{i=0}^{j} \tau_i'(x,r,R) \leq t\right\}. \tag{2.4}$$

[4] If the starting point x_0 lies inside $B(x,R)$, then the Brownian motion may travel from $\partial B(x,r)$ to $\partial B(x,R)$ before time T_0. To simplify the application of Dembo, Peres and Rosen [9, Lemma 2.4], we do not call this an excursion from $\partial B(x,r)$ to $\partial B(x,R)$.

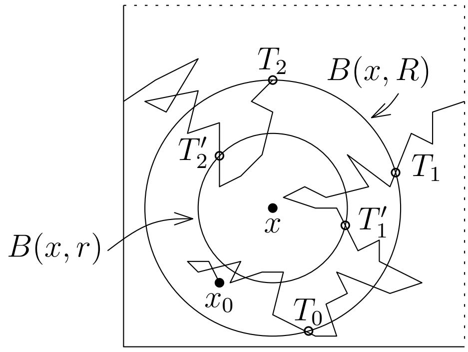

Figure 3: Hittings that define the times T_i, $i \in \mathbb{N}_0$, and T_i', $i \in \mathbb{N}$. The open circles indicate the locations of the starting and ending points $\xi_i'(x) = W(T_i')$, $\xi_i(x) = W(T_i)$ of the excursions.

Thus, $N(x,t,r,R)$ is the number of completed excursions from $\partial B(x,r)$ to $\partial B(x,R)$ by time t, while $N'(x,t,r,R)$ is the number of (necessarily completed) excursions when the total time spent *not* making an excursion reaches t.

As we will see in Proposition 2.1 below, $N(x,t,r,R)$ and $N'(x,t,r,R)$ have very similar scaling bahaviour for $t \to \infty$ and $r \ll R \ll 1$. Indeed, the times $\tau_i(x,r,R)$ and $\tau_i'(x,r,R)$ are typically large (since the Brownian motion typically visits the bulk of $\mathbb{T}^d$ many times before travelling from $\partial B(x,R)$ to $\partial B(x,r)$), whereas $\tau_i(x,r,R) - \tau_i'(x,r,R) = T_i(x) - T_i'(x)$ scales as R^2. The advantage of $N'(x,t,r,R)$ is that it is independent of hitting events within $B(x,r)$ given the starting and ending points $\xi_i'(x)$,$\xi_i(x)$ of the excursions.

Define

$$N_d(t,r,R) = \frac{\kappa_d t}{r^{-(d-2)} - R^{-(d-2)}}. \tag{2.5}$$

The following proposition shows that $N_d(t,r,R)$ represents the typical size for the random variables $N(x,t,r,R)$ and $N'(x,t,r,R)$.

Proposition 2.1. *For any $\delta \in (0,1)$ there is a $c = c(\delta) > 0$ such that, uniformly in $x, x_0 \in \mathbb{T}^d$, $t > 1$ and $0 < r^{1-\delta} \le R \le c$,*

$$\mathbb{P}_{x_0}\big(N(x,t,r,R) \ge (1+\delta)N_d(t,r,R)\big) \le e^{-cN_d(t,r,R)}, \tag{2.6}$$

$$\mathbb{P}_{x_0}\big(N'(x,t,r,R) \ge (1+\delta)N_d(t,r,R)\big) \le e^{-cN_d(t,r,R)}, \tag{2.7}$$

$$\mathbb{P}_{x_0}\big(N(x,t,r,R) \le (1-\delta)N_d(t,r,R)\big) \le e^{-cN_d(t,r,R)}. \tag{2.8}$$

Proof. The result follows from a lemma in Dembo, Peres and Rosen [9], which we reformulate in our notation. (Note that the constant κ_d defined by (1.4) corresponds to the quantity $1/\kappa_{\mathbb{T}^d}$ from [9, page 2] rather than $\kappa_{\mathbb{T}^d}$.)

Lemma 2.2 ([9, Lemma 2.4])**.** *There is a constant $\eta > 0$ such that if $N \ge \eta^{-1}$, $0 < \delta < \delta_0 < \eta$ and $0 < 2r \le R < R_0(\delta)$, then for some $c = c(r,R) > 0$ and uniformly*

in $x, x_0 \in \mathbb{T}^d$,

$$\mathbb{P}_{x_0}\left(1 - \delta \leq \frac{\kappa_d}{N\big(r^{-(d-2)} - R^{-(d-2)}\big)} \sum_{i=0}^{N} \tau_i(x, r, R) \leq 1 + \delta\right) \geq 1 - e^{-c\delta^2 N}. \tag{2.9}$$

Moreover, c *can be chosen to depend only on* δ_0 *as soon as* $R > r^{1-\delta_0}$*. The same result holds when* $\tau_i'(x, r, R)$ *is replaced by* $\tau_i(x, r, R)$*.*

(The same result for τ_i' is not included in [9], but follows from the estimates in that paper. Indeed, $\tau_i - \tau_i'$ is shown to be an error term.)

To prove Proposition 2.1, we begin with (2.8). Fix $\delta > 0$. We may assume without loss of generality that $\delta < \frac{1}{2}$ and $1/(1 - \frac{1}{2}\delta) < 1 + \frac{2}{3}\delta < 1 + \eta$. Set $N = \lfloor (1-\delta) N_d(t, r, R) \rfloor + 1$. Since $N/N_d(t, r, R) \to 1 - \delta$ as $N_d(t, r, R) \to \infty$, we can choose r small enough so that $\frac{1}{2} N_d(t, r, R) \leq N \leq (1 - \frac{1}{2}\delta) N_d(t, r, R)$ and $N \geq \eta^{-1}$, uniformly in R and $t > 1$. We have

$$\big\{N(x, t, r, R) \leq (1-\delta) N_d(t, r, R)\big\} = \{N(x, t, r, R) < N\} = \{T_N \geq t\}. \tag{2.10}$$

Since $T_N = \sum_{i=0}^N \tau_i(x, r, R)$, it follows that

$$\begin{aligned}
&\mathbb{P}_{x_0}\big(N(x, t, r, R) \leq (1-\delta) N_d(t, r, R)\big) \\
&\quad = \mathbb{P}_{x_0}\left(\sum_{i=0}^{N} \tau_i(x, r, R) \geq t\right) = \mathbb{P}_{x_0}\left(\frac{\kappa_d \sum_{i=0}^N \tau_i(x, r, R)}{N\big(r^{-(d-2)} - R^{-(d-2)}\big)} \geq \frac{N_d(t, r, R)}{N}\right) \\
&\quad \leq \mathbb{P}_{x_0}\left(\frac{\kappa_d \sum_{i=0}^N \tau_i(x, r, R)}{N\big(r^{-(d-2)} - R^{-(d-2)}\big)} \geq \frac{1}{1 - \frac{1}{2}\delta}\right).
\end{aligned} \tag{2.11}$$

Hence (2.8) follows from Lemma 2.2 with δ and δ_0 replaced by $\frac{1}{2}\delta/(1 - \frac{1}{2}\delta)$ and $\frac{2}{3}\delta$, respectively, with the constant c in Proposition 2.1 chosen small enough so that $2r \leq R < R_0[\frac{1}{2}\delta/(1 - \frac{1}{2}\delta)]$.

The proof of (2.7) is similar. Let $\delta > 0$ be such that $\frac{1}{2}\delta/(1 + \frac{1}{2}\delta) < \eta$ and set $N' = \lceil (1+\delta) N_d(t, r, R) \rceil$. As before, we have

$$\mathbb{P}_{x_0}\big(N'(t, x, r, R) \geq (1+\delta) N_d(t, r, R)\big) \leq \mathbb{P}_{x_0}\Big(\frac{\kappa_d \sum_{i=0}^{N'} \tau_i'(x, r, R)}{N'\big(r^{-(d-2)} - R^{-(d-2)}\big)} \leq \frac{1}{1 + \frac{1}{2}\delta}\Big) \tag{2.12}$$

and we can apply the version of Lemma 2.2 with $\tau_i'(x, r, R)$ instead of $\tau_i(x, r, R)$ and δ replaced by $\frac{1}{2}\delta/(1 + \frac{1}{2}\delta)$.

Finally, because $N'(t, x, r, R) \leq N(t, x, r, R)$, (2.6) follows from (2.7). $\square$

Proposition 2.1 forms the link between the global structure of $\mathbb{T}^d$, notably the fact that a Brownian motion on $\mathbb{T}^d$ has a finite mean return time to a small ball, and the excursions of W within small balls, during which W cannot be distinguished from a Brownian motion on all of $\mathbb{R}^d$.

2.2 Hitting sets by excursions

The concentration inequalities in Proposition 2.1 will allow us to treat the number of excursions as deterministic. This observation motivates the following definition.

Definition 2.3. Let $0 < r < R < \frac{1}{2}$, $\varphi > 0$ and $N \in \mathbb{N}$. A pair (x, E) with $x \in \mathbb{T}^d$, $E \subset \mathbb{R}^d$ Borel, will be called (N, φ, r, R)-*successful* if none of the first N excursions of W from $\partial B(x, r)$ to $\partial B(x, R)$ hit $x + \varphi E$.

Proposition 2.4. *Let $0 < \epsilon < r < R < \frac{1}{2}$. Then, uniformly in $\varphi > 0$, $x_0, x \in \mathbb{T}^d$ and $E \subset \mathbb{R}^d$ a Borel set with $\varphi E \subset B(0, \epsilon)$, and uniformly in $(\xi_i'(x), \xi_i(x))_{i=1}^N$,*

$$\begin{aligned}\mathbb{P}_{x_0}\big((x, E) \text{ is } (N, \varphi, r, R)\text{-successful} \,\big|\, (\xi_i'(x), \xi_i(x))_{i=1}^N\big) \\ = \exp\left[-N\left(\frac{\varphi}{r}\right)^{d-2} \frac{\operatorname{Cap} E}{\kappa_d}[1 + o(1)]\right], \qquad r/\epsilon, R/r \to \infty.\end{aligned} \tag{2.13}$$

Since the error term is uniform in $(\xi_i'(x), \xi_i(x))_{i=1}^N$, Proposition 2.4 also applies to the unconditional probability $\mathbb{P}_{x_0}((x, E)$ is (N, φ, r, R)-successful$)$.

To prove Proposition 2.4 we need the following lemma for the hitting probability of a single excursion given its starting and ending points. For $\xi' \in \partial B(x, r)$, $\xi \in \partial B(x, R)$, write $\mathbb{P}_{\xi',\xi}$ for the law of an excursion $W[0, \zeta_R]$, $\zeta_R = \inf\{t \geq 0\colon\ d(x, W(t)) \geq R\}$, from $\partial B(x, r)$ to $\partial B(x, R)$, started at ξ' and conditioned to end at ξ.

Lemma 2.5. *Let $0 < \epsilon < r < R < \frac{1}{2}$. Then, uniformly in $x \in \mathbb{T}^d$, $\xi' \in \partial B(x, r), \xi \in \partial B(x, R)$ and E a Borel set with $E \subset B(0, \epsilon)$,*

$$\mathbb{P}_{\xi',\xi}((x + E) \cap W[0, \zeta_R] \neq \varnothing) = \frac{\operatorname{Cap} E}{\kappa_d\, r^{d-2}}\,[1 + o(1)], \qquad r/\epsilon, R/r \to \infty. \tag{2.14}$$

Lemma 2.5 is a more elaborate version of (1.10): it states that the asymptotics of (1.10) remain valid when we stop the Brownian motion upon exiting a sufficiently distant ball, and hold conditionally and uniformly, provided the balls and set are well separated. In the proof we use the relation

$$\int_{\partial B(0,r)} \mathbb{P}_x(E \cap W\,[0, \infty) \neq \varnothing)\, d\sigma_r(x) = \frac{\operatorname{Cap} E}{\kappa_d\, r^{d-2}}, \qquad E \text{ a Borel subset of } B(0, r), \tag{2.15}$$

where σ_r denotes the uniform measure on $\partial B(0, r)$. Equation (2.15) becomes an identity as soon as $B(0, r)$ contains E, and as such it is a more precise version of (1.10): see Port and Stone [18, Chapter 3, Theorem 1.10] and surrounding material.

We defer the proof of Lemma 2.5 to Section A.1. We can now prove Proposition 2.4.

Proof. Conditional on their starting and ending points $(\xi_i'(x), \xi_i(x))_{i=1}^N$, the successive excursions from $\partial B(x, r)$ to $\partial B(x, R)$ are independent with laws $\mathbb{P}_{\xi_i'(x),\xi_i(x)}$. Applying Lemma 2.5, we have

$$\begin{aligned}&\mathbb{P}_{x_0}\big((x, E) \text{ is } (N, \varphi, r, R)\text{-successful} \,\big|\, (\xi_i'(x), \xi_i(x))_{i=1}^N\big) \\ &= \prod_{i=1}^N \mathbb{P}_{\xi_i'(x),\xi_i(x)}((x + \varphi E) \cap W[0, \zeta_R] = \varnothing) = \left(1 - \frac{\operatorname{Cap}(\varphi E)}{\kappa_d\, r^{d-2}}\,[1 + o(1)]\right)^N.\end{aligned} \tag{2.16}$$

Since $\mathrm{Cap}(\varphi E) \le \kappa_d\, \epsilon^{d-2} = o(r^{d-2})$ as $r/\epsilon \to \infty$, we can rewrite the right-hand side of (2.16) as

$$\exp\left[-N\frac{\mathrm{Cap}(\varphi E)}{\kappa_d\, r^{d-2}}\,[1+o(1)]\right], \tag{2.17}$$

so that the scaling relation in (1.8) implies the claim. □

2.3 Properties of capacity

In this section we collect a few elementary properties of capacity.

2.3.1 Continuity

Proposition 2.6. *Let E denote a Borel subset of $\mathbb{R}^d$.*

(a) If E is compact, then $\mathrm{Cap}\, E_r \downarrow \mathrm{Cap}\, E$ as $r \downarrow 0$.

(b) If E is open, then $\mathrm{Cap}\, E_{-r} \uparrow \mathrm{Cap}\, E$ as $r \downarrow 0$.

(c) If E is bounded with $\mathrm{Cap}(\mathrm{clo}(E)) = \mathrm{Cap}(\mathrm{int}(E))$, then $\mathrm{Cap}\, E_r \downarrow \mathrm{Cap}\, E$ and $\mathrm{Cap}\, E_{-r} \uparrow \mathrm{Cap}\, E$ as $r \downarrow 0$.

Proof. For $r \downarrow 0$ we have $E_r \downarrow \mathrm{clo}(E)$ and $E_{-r} \uparrow \mathrm{int}(E)$ for any set E. By Port and Stone [18, Chapter 3, Proposition 1.13], it follows that $\mathrm{Cap}\, E_{-r} \uparrow \mathrm{Cap}(\mathrm{int}(E))$ and, if E is bounded, $\mathrm{Cap}\, E_r \downarrow \mathrm{Cap}(\mathrm{clo}(E))$. The statements about E follow depending on which inequalities in $\mathrm{Cap}(\mathrm{int}(E)) \le \mathrm{Cap}\, E \le \mathrm{Cap}(\mathrm{clo}(E))$ are equalities. □

Proposition 2.6 is a statement about the continuity of $E \mapsto \mathrm{Cap}\, E$ with respect to enlargement and shrinking. The assumptions on E are necessary, since there are sets E with $\mathrm{Cap}(\mathrm{clo}(E)) > \mathrm{Cap}(\mathrm{int}(E))$. Note that $E \mapsto \mathrm{Cap}\, E$ is *not* continuous with respect to the Hausdorff metric, even when restricted to reasonable classes of sets. For instance, the finite sets $B(0,1) \cap \frac{1}{n}\mathbb{Z}^d$ converge to $B(0,1)$ in the Hausdorff metric, but have zero capacity for all n.

2.3.2 Separation

Lemma 2.7. *Let $0 < \epsilon < r$. Then, uniformly in $x_1, x_2 \in \mathbb{R}^d$ with $d(x_1,x_2) \ge r$ and $E^{(1)}, E^{(2)}$ Borel subsets of $\mathbb{R}^d$ with $E^{(1)}, E^{(2)} \subset B(0,\epsilon)$,*

$$\mathrm{Cap}\left((x_1+E^{(1)})\cup(x_2+E^{(2)})\right) = \left(\mathrm{Cap}\, E^{(1)}+\mathrm{Cap}\, E^{(2)}\right)[1-o(1)], \qquad r/\epsilon \to \infty. \tag{2.18}$$

Proof. Fix $\tilde r$ large enough so that $(x_1 + E^{(1)}) \cup (x_2 + E^{(2)}) \subset B(0, \tilde r)$. On the event $\{W \text{ hits } x_j + E^{(j)}\}$, write Y_j for the first point of $x_j + E^{(j)}$ hit by W. Applying (1.9),

(2.15), and the Markov property, we get

$$
\begin{aligned}
0 &\le \operatorname{Cap}(x_1 + E^{(1)}) + \operatorname{Cap}(x_2 + E^{(2)}) - \operatorname{Cap}\big((x_1 + E^{(1)}) \cup (x_2 + E^{(2)})\big) \\
&= \kappa_d\, \tilde{r}^{d-2} \int_{\partial B(0,\tilde{r})} \mathbb{P}_x\big(W \text{ hits } x_1 + E^{(1)} \text{ and } x_2 + E^{(2)}\big)\, d\sigma_{\tilde{r}}(x) \\
&\le \sum_{\{j,j'\}=\{1,2\}} \kappa_d\, \tilde{r}^{d-2} \int_{\partial B(0,\tilde{r})} \mathbb{E}_x\Big(\mathbb{1}_{\{W \text{ hits } x_j + E^{(j)}\}} \mathbb{P}_{Y_j}\big(W \text{ hits } B(x_{j'}, \epsilon)\big)\Big)\, d\sigma_{\tilde{r}}(x) \\
&\le \sum_{\{j,j'\}=\{1,2\}} \kappa_d\, \tilde{r}^{d-2} \int_{\partial B(0,\tilde{r})} \mathbb{P}_x\big(W \text{ hits } x_j + E^{(j)}\big)\, \frac{\epsilon^{d-2}}{(r-\epsilon)^{d-2}}\, d\sigma_{\tilde{r}}(x) \\
&= \frac{\epsilon^{d-2}}{(r-\epsilon)^{d-2}} \left(\operatorname{Cap} E^{(1)} + \operatorname{Cap} E^{(2)}\right), \qquad (2.19)
\end{aligned}
$$

where the second inequality uses that every $Y_j \in x_j + E^{(j)}$ is at least a distance $r - \epsilon$ from $x_{j'}$. But $(\epsilon/(r-\epsilon))^{d-2} = o(1)$ for $r/\epsilon \downarrow 0$, and so the claim follows. $\square$

3 Hitting probabilities of lattice animals

An event such as

$$\big\{\exists x \in \mathbb{T}^d \colon\ (x + \varphi_d(t)E) \cap W[0,t] = \varnothing\big\} \qquad (3.1)$$

is a simultaneous statement about an infinite collection $(x + \varphi_d(t)E)_{x \in \mathbb{T}^d}$ of sets. In this section, we apply the results of Section 2 to prove simultaneous statements for a finite collection of discretized sets. Section 3.1 proves a bound for sets of large capacity that forms the basis for Theorem 1.1, while Section 3.2 does the same for sets of small capacity that form the basis for Theorems 1.2–1.3.

Definition 3.1. A *lattice animal* is a connected set $A \subset \mathbb{R}^d$ that is the union of a finite number of closed unit cubes with centres in $\mathbb{Z}^d$. We write $\mathcal{A}^\square$ for the collection of all lattice animals, and $\mathcal{A}^\square_Q$ for the collection of lattice animals $A \in \mathcal{A}^\square$ that contain 0 and consist of at most Q unit cubes.

It is readily verified that, for any $d \ge 2$, there is a constant $C < \infty$ such that

$$\left|\mathcal{A}^\square_Q\right| \le e^{CQ}, \qquad Q \in \mathbb{N}. \qquad (3.2)$$

In fact, subadditivity arguments show that $\left|\mathcal{A}^\square_Q\right|$ grows exponentially, in the sense that $\lim_{Q\to\infty} \left|\mathcal{A}^\square_Q\right|^{1/Q}$ exists in $(1,\infty)$ for any $d \ge 2$. See, for instance, Klarner [12] for the case $d = 2$, or Mejia Miranda and Slade [15, Lemma 2] for a general upper bound that implies (3.2).

Lattice animals are commonly considered as discrete combinatorial objects. In our context, we can identify $A \in \mathcal{A}^\square$ with the collection $A \cap \mathbb{Z}^d$ of lattice points in A. Requiring A to be a connected subset of $\mathbb{R}^d$ is then equivalent to requiring the vertices $A \cap \mathbb{Z}^d$ to form a connected subgraph of the lattice $\mathbb{Z}^d$. (Because of the details of our

definition, the relevant choice of lattice structure is that vertices $x, y \in \mathbb{Z}^d$ are adjacent when their ℓ_∞-distance is 1.)

For $n \in \mathbb{N}$, set $G_n = x + \frac{1}{n}\mathbb{Z}^d$ to be a *grid* of n^d points in $\mathbb{T}^d$, for some $x \in \mathbb{T}^d$. The choice of x (i.e., the alignment of the grid) will generally not be relevant to our purposes.

3.1 Hitting large lattice animals

Proposition 3.2. *Fix an integer-valued function $t \mapsto n(t)$ such that*

$$\lim_{t\to\infty} \frac{n(t)\varphi_d(t)}{(\log t)^{1/d}} = 0. \tag{3.3}$$

Given $A \in \mathcal{A}^\square$, write $E(A) = n(t)^{-1}\varphi_d(t)^{-1}A$. Then, for each κ,

$$\limsup_{t\to\infty} \frac{\log \mathbb{P}_{x_0}\big(\exists x \in G_{n(t)}, A \in \mathcal{A}^\square\colon\ \operatorname{Cap} E(A) \geq \kappa, (x + \varphi_d(t)E(A)) \cap W[0,t] = \varnothing\big)}{\log t} \leq J_d(\kappa). \tag{3.4}$$

Proposition 3.2 gives an upper bound on the probability of finding unhit sets of large capacity, simultaneously over all sets of the form $E(A)$, $A \in \mathcal{A}^\square$. Note that $x + \varphi E(A)$ is a finite union of cubes of side length $1/n$ centred at points of G_n. In Section 4 we will use $x + \varphi E(A)$ as a lattice approximation to a generic set $x + \varphi E$. The fineness of this lattice approximation is determined by the relation between the lengths $1/n$ and φ. The hypothesis in (3.3) means that the lattice scale $1/n$ is a factor of order $o((\log t)^{1/d})$ smaller compared to the scale φ. This order is chosen so that the number of lattice animals does not grow too quickly.

Before proving Proposition 3.2, we give some definitions and make some remarks that we will use throughout Section 3. We abbreviate

$$\varphi = \varphi_d(t), \qquad n = n(t), \qquad E(A) = n^{-1}\varphi^{-1}A. \tag{3.5}$$

For $x \in \mathbb{T}^d$, we introduce the nested balls $B(x, r)$ and $B(x, R)$, where

$$r = \varphi^{1-\delta}, \qquad R = \varphi^{1-2\delta}, \tag{3.6}$$

and $\delta \in (0, \frac{1}{2})$ is fixed. We have $\varphi \ll r \ll R \to 0$ as $t \to \infty$, and we will always take t large enough so that $\varphi < 1$ and $R < \frac{1}{2}$.

Suppose that $A \in \mathcal{A}^\square$ is such that $\operatorname{Cap} E(A)$ is bounded. By (1.44), it follows that $\operatorname{Vol} E(A)$ is also bounded. Consequently, we may assume that A consists of at most $Q = Q(t)$ unit cubes, where Q is suitably chosen with

$$Q = O(n^d\varphi^d). \tag{3.7}$$

Suppose, instead, that $A \in \mathcal{A}^\square$ is minimal subject to the condition $\operatorname{Cap} E(A) \geq \kappa$, and suppose that $n\varphi \to \infty$. By (1.9), upon removing a single unit cube from A the capacity

$\operatorname{Cap} E(A)$ decreases by at most $O(1/n\varphi)$, and so it follows that $\kappa \leq \operatorname{Cap} E(A) \leq \kappa + O(1/n\varphi)$. In particular, $\operatorname{Cap} E(A)$ is bounded, and we may again assume (3.7).

Given $x \in G_n$ and $A \in \mathcal{A}^\square$, the translate $x + \varphi E(A)$ can always be written as $x' + \varphi E(A')$, where $x' \in G_n$ and $0 \in A'$. By the above, we have $A' \in \mathcal{A}^\square_Q$. Since A' is connected and $0 \in A'$, it follows that $\varphi E(A') \subset B(0, \varphi Q\sqrt{d})$. If $Q = t^{o(1)}$ (in particular, if (3.3) is assumed, or the weaker hypothesis in (3.16)), then $r/\varphi Q \to \infty$ as $t \to \infty$. We may therefore always take t large enough so that $B(0, \varphi Q\sqrt{d}) \subset B(0, r)$, and we may apply Proposition 2.4 to $\varphi E(A)$, uniformly over $A \in \mathcal{A}^\square_Q$.

Proof. Note that if we replace n by a suitable multiple $kn = k(t)n(t)$ for $k(t) \in \mathbb{N}$, we can only increase the probability in (3.4). Thus it is no loss of generality to assume that $n\varphi \to \infty$.

The event that W hits $x + \varphi E(A)$ is decreasing in A. Therefore we may restrict our attention to lattice animals A that are minimal subject to $\operatorname{Cap} E(A) \geq \kappa$. By the remarks above, we may assume that $A \in \mathcal{A}^\square_Q$. Combining (3.3) and (3.7), we have $Q = o(\log t)$.

Set $N = (1-\delta)N_d(t, r, R)$. Recalling (2.5) and (3.6), we have $N_d(t, r, R) = t^{\delta + o(1)}$ as $t \to \infty$. If the event in (3.4) occurs, then there must exist a point $x \in G_n$ with $N(x, t, r, R) < N$ or a pair $(x, A) \in G_n \times \mathcal{A}^\square_Q$ such that $\operatorname{Cap} E(A) \geq \kappa$ and $(x, E(A))$ is $(\lfloor N \rfloor, \varphi, r, R)$-successful. Write $\tilde{\chi}^\square$ for the number of such pairs. Then

$$\begin{aligned}
&\mathbb{P}_{x_0}\big(\exists x \in G_n, A \in \mathcal{A}^\square\colon\ \operatorname{Cap} E(A) \geq \kappa, (x + \varphi E(A)) \cap W[0,t] = \varnothing\big) \\
&\quad \leq |G_n| \max_{x \in G_n} \mathbb{P}_{x_0}(N(x, t, r, R) < N) + \mathbb{P}_{x_0}(\tilde{\chi}^\square \geq 1) \\
&\quad \leq t^{d/(d-2)+o(1)} e^{-ct^{\delta+o(1)}} + \mathbb{P}_{x_0}(\tilde{\chi}^\square \geq 1) \qquad (3.8)
\end{aligned}$$

by Proposition 2.1. The first term in the right-hand side is negligible. For the second term, $Q = o(\log t)$ implies that $\left|\mathcal{A}^\square_Q\right| \leq e^{O(Q)} = t^{o(1)}$ by (3.2), and so Proposition 2.4 gives

$$\begin{aligned}
\mathbb{E}(\tilde{\chi}^\square) &\leq |G_n| \left|\mathcal{A}^\square_Q\right| \max_{x \in G_n, A \in \mathcal{A}^\square_Q} \mathbb{P}_{x_0}((x, E(A)) \text{ is } (\lfloor N \rfloor, \varphi, r, R)\text{-successful}) \\
&\leq (t^{d/(d-2)+o(1)})(t^{o(1)})(t^{-d\kappa/[(d-2)\kappa_d]+O(\delta)}) \leq t^{-d(\kappa/\kappa_d - 1)/(d-2)+O(\delta)}, \qquad (3.9)
\end{aligned}$$

and the Markov inequality completes the proof. □

Proposition 3.2 bounds the probability that a single rescaled lattice animal $x + \varphi_d(t)E(A)$ is not hit. We will also need the following bounds, for finite unions of lattice animals that are relatively close, and for pairs of lattice animals that are relatively distant.

Lemma 3.3. *Assume* (3.3). *Fix a capacity $\kappa \geq \kappa_d$, a positive integer $k \in \mathbb{N}$ and a positive function $t \mapsto h(t) > 0$ satisfying*

$$\lim_{t \to \infty} \frac{\log(h(t)/\varphi_d(t))}{\log t} = 0. \qquad (3.10)$$

Then the probability that there exist a point $x \in G_{n(t)}$ and lattice animals $A^{(1)}, \dots, A^{(k)} \in \mathcal{A}^\square$, such that the union $E = \cup_{j=1}^k E(A^{(j)})$ satisfies $\operatorname{Cap} E \geq \kappa$, $\varphi_d(t)E \subset B(0, h(t))$, and $(x + \varphi_d(t)E) \cap W[0,t] = \varnothing$, is at most $t^{-I_d(\kappa)+o(1)}$.

Proof. The proof is the same as for Proposition 3.2. Abbreviate $h = h(t)$. Since $h = t^{o(1)}\varphi$, it follows that $r/h \to \infty$ as $t \to \infty$, so that Proposition 2.4 applies to φE. Similarly, writing $A^{(j)} = y_j + \tilde{A}^{(j)}$ with $\tilde{A}^{(j)} \in \mathcal{A}_Q^\square$ and $y_j \in B(0, nh) \cap \mathbb{Z}^d$, we have that there are at most $O((nh)^{dk}) \left|\mathcal{A}_Q^\square\right|^k$ possible choices for $A^{(1)}, \dots, A^{(k)}$. This number is $t^{o(1)}$ by (3.3) and (3.10), so that a counting argument applies as before. $\square$

Lemma 3.4. *Assume* (3.3)*. Fix a positive function $t \mapsto h(t) > 0$ satisfying*

$$\liminf_{t\to\infty} \frac{h(t)}{\varphi_d(t) \log t} > 0, \tag{3.11}$$

and let $\kappa^{(1)}, \kappa^{(2)} > \kappa_d$, $x_1 \in \mathbb{T}^d$. Then the probability that there exist a point $x_2 \in G_{n(t)}$ with $d(x_1, x_2) \geq h(t)$ and lattice animals $A^{(1)}, A^{(2)} \in \mathcal{A}^\square$ with $\operatorname{Cap} E(A^{(j)}) \geq \kappa^{(j)}$ such that $(x_j + \varphi_d(t)E(A^{(j)})) \cap W[0,t] = \varnothing$, $j = 1, 2$, is at most $t^{-[d\kappa^{(1)}/(d-2)\kappa_d] - I_d(\kappa^{(2)})+o(1)}$.

Proof. We resume the notation and assumptions from the proof of Proposition 3.2, this time taking $\delta < \frac{1}{4}$. Abbreviate $h = h(t)$.

For $x_2 \in G_n$ such that $d(x_1, x_2) \geq 2R$, the events of $(x_j, E(A_j))$ being $(\lfloor N \rfloor, \varphi, r, R)$-successful, $j = 1, 2$, are conditionally independent given $(\xi_i'(x_j), \xi_i(x_j))_{i,j}$. The required bound for the case $d(x_1, x_2) \geq 2R$ therefore follows by the same argument as in the proof of Proposition 3.2.

For $x_2 \in G_n$ such that $d(x_1, x_2) \leq 2R$, set $\tilde{r} = \varphi^{1-3\delta}$, $\tilde{R} = \varphi^{1-4\delta}$ and $\tilde{N} = (1 - \delta)N_d(t, \tilde{r}, \tilde{R})$. We have $\varphi E(A_j) \subset B(0, \varphi Q\sqrt{d})$ for $j = 1, 2$, with $Q = o(\log t)$ (without loss of generality, as in the proof of Proposition 3.2). Write $x_2 = x_1 + \varphi y$, where $y \in \mathbb{R}^d$ with $h/\varphi \leq d(0, y) \leq 2R/\varphi$. The hypothesis (3.11) implies that $h/\varphi Q \to \infty$. Hence we can apply Lemma 2.7 (with $\epsilon = \varphi Q\sqrt{d}$ and h playing the role of r), to conclude that

$$\operatorname{Cap}\big(E(A_1) \cup (y + E(A_2))\big) = \big(\operatorname{Cap} E(A_1) + \operatorname{Cap} E(A_2)\big)[1 - o(1)]. \tag{3.12}$$

We also have $E(A_1) \cup (y + E(A_2)) \subset B(0, 2R + \varphi Q\sqrt{d})$ with $\tilde{r}/R, \tilde{r}/\varphi Q \to \infty$. In particular, $x_1 + \varphi(E(A_1) \cup (y + E(A_2))) \subset B(x_1, \tilde{r})$ for t large enough. As in the proof of Proposition 3.2, $(x_j + \varphi E(A_j)) \cap W[0,t] = \varnothing$ implies that $N(x_1, t, \tilde{r}, \tilde{R}) < N$ or $(x_1, E(A_1) \cup (y + E(A_2)))$ is $(\lfloor \tilde{N} \rfloor, \varphi, \tilde{r}, \tilde{R})$-successful. By (3.12) and Proposition 2.4,

$$\begin{aligned}\mathbb{P}_{x_0}\Big(\big(x_1, E(A_1) \cup (y + E(A_2))\big) \text{ is } (\lfloor \tilde{N} \rfloor, \varphi, \tilde{r}, \tilde{R})\text{-successful}\Big)& \\ \leq \exp\Big[-\tilde{N}(\varphi/\tilde{r})^{d-2}(\kappa^{(1)} + \kappa^{(2)} - o(1))/\kappa_d\Big],&\end{aligned} \tag{3.13}$$

and the rest of the proof is the same as for Proposition 3.2. $\square$

3.2 Hitting small lattice animals

The bound in Proposition 3.2 is only meaningful when $\kappa > \kappa_d$. For $\kappa < \kappa_d$, there are likely to be many unhit sets of capacity κ. The next two propositions quantify this statement.

Proposition 3.5. *Fix an integer-valued function $t \mapsto n(t)$ satisfying condition (3.3) such that $\lim_{t\to\infty} n(t)\varphi_d(t) = \infty$. For $E \subset \mathbb{R}^d$, write $\chi(t, n(t), E)$ for the number of points $x \in G_{n(t)}$ such that $(x + \varphi_d(t)E) \cap W[0,t] = \varnothing$, and write $\chi^{\text{disjoint}}(t, n(t), E)$ for the maximal number of disjoint translates $x + \varphi_d(t)E$ such that $x \in G_{n(t)}$ and $(x + \varphi_d(t)E) \cap W[0,t] = \varnothing$. For $\kappa > 0$, define*

$$\begin{aligned} \chi_+^\square(t, n(t), \kappa) &= \sum_{\substack{A\in\mathcal{A}^\square:\ 0\in A,\\ \operatorname{Cap} E(A)\geq\kappa}} \chi(t, n(t), E(A)), \\ \chi_-^\square(t, n(t), \kappa) &= \min_{\substack{A\in\mathcal{A}^\square:\\ \operatorname{Cap} E(A)\leq\kappa}} \chi^{\text{disjoint}}(t, n(t), E(A)). \end{aligned} \tag{3.14}$$

Then, for $0 < \kappa < \kappa_d$,

$$\lim_{t\to\infty} \frac{\log \chi_+^\square(t, n(t), \kappa)}{\log t} = J_d(\kappa), \lim_{t\to\infty} \frac{\log \chi_-^\square(t, n(t), \kappa)}{\log t} = J_d(\kappa), \quad \textit{in } \mathbb{P}_{x_0}\textit{-probability.} \tag{3.15}$$

Proposition 3.6. *Assume the hypotheses of Theorem 1.3, and fix an integer-valued function $t \mapsto n(t)$ such that*

$$\lim_{t\to\infty} \frac{\log[n(t)\varphi_d(t)]}{\log t} = 0. \tag{3.16}$$

Given $A \in \mathcal{A}^\square$, write $E(A) = n(t)^{-1}\varphi_d(t)^{-1}A$. Then, for each $\kappa \in (0, \kappa_d)$,

$$\mathbb{P}_{x_0}\big(\exists A \in \mathcal{A}^\square:\ \operatorname{Cap} E(A) \leq \kappa \textit{ and } (x + \varphi_d(t)E(A)) \cap W[0,t] \neq \varnothing\ \forall x \in S(t)\big) \leq \exp\left[-t^{J_d(\kappa)-o(1)}\right]. \tag{3.17}$$

In Proposition 3.6, the scale of the lattice need only satisfy (3.16) instead of the stronger condition (3.3). This reflects the difference in scaling between the probabilities in Proposition 3.6 compared to Proposition 3.2.

3.2.1 Proof of Proposition 3.5

Proof. Let $\delta \in (0, \frac{1}{2})$ be given. It suffices to show that $t^{J_d(\kappa)-O(\delta)} \leq \chi_-^\square(t, n, \kappa)$ and $\chi_+^\square(t, n, \kappa) \leq t^{J_d(\kappa)+O(\delta)}$ with high probability. (Given $\kappa < \kappa'$, the assumption $n\varphi \to \infty$ implies the existence of some A with $\kappa \leq \operatorname{Cap} E(A) \leq \kappa'$, and therefore $\chi_-^\square(t, n, \kappa') \leq \chi_+^\square(t, n, \kappa)$.)

For the upper bound, recall N and $\tilde{\chi}^\square$ from the proof of Proposition 3.2. On the event $\{N(x, t, r, R) < N\ \forall x \in G_n\}$ (whose probability tends to 1) we have $\chi_+^\square(t, \kappa, n) \leq \tilde{\chi}^\square$. From (3.9) it follows that $\tilde{\chi}^\square \leq t^{J_d(\kappa)+O(\delta)}$ with high probability.

For the lower bound, let $\{x_1, \dots, x_K\}$ denote a maximal collection of points in G_n satisfying $d(x_j, x_k) > 2R$ for $j \neq k$, so that $K = R^{-d+o(1)} = t^{d/(d-2)-O(\delta)}$. Write $N_- = (1+\delta)N_d(t, r, R)$. By Proposition 2.1, in the same way as in the proof of Proposition 3.2, $N(x_j, t, r, R) \leq N_-$ for each $j = 1, \dots, K$, with high probability. Moreover we may take t large enough so that $\varphi E(A) \subset B(0, R)$, so that the translates $x_j + \varphi E(A)$ are disjoint. Let $\tilde{\chi}_-^{\square}(A)$ denote the number of points x_j, $j \in \{1, \dots, K\}$, such that $(x_j, E(A))$ is $(\lceil N_- \rceil, \varphi, r, R)$-successful. We have $\chi_-^{\square}(t, n(t), E(A)) \geq \tilde{\chi}_-^{\square}(A) - 1$ on the event $\{N(x_j, t, r, R) \leq N_- \ \forall j\}$, since at most one translate $x_j + \varphi E(A)$ may have been hit before the start of the first excursion, in the case $x_0 \in B(x_j, R)$. On the other hand, since the balls $B(x_j, R)$ are disjoint, the excursions are conditionally independent given the starting and ending points $(\xi_i'(x_j), \xi_i(x_j))_{i,j}$. It follows that, for each A with $\operatorname{Cap} E(A) \leq \kappa$, $\tilde{\chi}_-^{\square}(A)$ is stochastically larger than a Binomial(K, p) random variable, where $p \geq t^{-d\kappa/(d-2)-O(\delta)}$ by Proposition 2.4. A straightforward calculation shows that $\mathbb{P}(\text{Binomial}(K, p) < \frac{1}{2}Kp) \leq e^{-cKp}$ for some $c > 0$, so that

$$\mathbb{P}_{x_0}(\tilde{\chi}_-^{\square}(A) \leq t^{J_d(\kappa)-O(\delta)}) \leq \exp\left[-ct^{J_d(\kappa)-O(\delta)}\right]. \tag{3.18}$$

As in the proof of Proposition 3.2, there are at most $t^{o(1)}$ animals A to consider, so a union bound completes the proof. $\square$

As with Lemma 3.3, we may modify Proposition 3.5 to deal with a finite union of lattice animals.

Lemma 3.7. *Assume the hypotheses of Proposition 3.5, let $k \in \mathbb{N}$, and let $t \mapsto h(t) > 0$ be a positive function satisfying* (3.10). *Define*

$$\begin{aligned} \chi_+^{\square}(t, n(t), \kappa, k, h(t)) &= \sum \chi(t, n(t), E), \\ \chi_-^{\square}(t, n(t), \kappa, k, h(t)) &= \min \chi^{\text{disjoint}}(t, n(t), E), \end{aligned} \tag{3.19}$$

where the sum and minimum are over sets $E = \cup_{j=1}^k E(A^{(j)})$ such that $\varphi_d(t)E \subset B(0, h(t))$; $(x + \varphi_d(t)E) \cap W[0, t] = \varnothing$; and $\operatorname{Cap} E \geq \kappa$ (for $\chi_+^{\square}$) or $\operatorname{Cap} E \leq \kappa$ (for $\chi_-^{\square}$), respectively. Then $(\log \chi_+^{\square}(t, n(t), \kappa, k, h(t)))/\log t$ and $(\log \chi_-^{\square}(t, n(t), \kappa, k, h(t)))/\log t$ converge in $\mathbb{P}_{x_0}$-probability to $J_d(\kappa)$ as $t \to \infty$.

3.2.2 Proof of Proposition 3.6

The proof of Proposition 3.5 compares $\chi_-^{\square}(t, n(t), \kappa)$ to a random variable that is approximately Binomial$(t^{d/(d-2)}, t^{-d\kappa/(d-2)})$. If this identification were exact, then the asymptotics in Proposition 3.6 would follow in a similar way. However, the bound for each individual probability $\mathbb{P}_{x_0}(N(x_j, t, r, R) \geq (1+\delta)N_d(t, r, R))$, $j = 1, \dots, K$, although relatively small, is still much larger than the probability in Proposition 3.6. Therefore an additional argument is needed.

Proof. Abbreviate $h = h(t), S = S(t)$.

Recall that the condition $\operatorname{Cap} E(A) \leq \kappa$ implies that A consists of at most Q cubes, where because of (3.7) and (3.16) we have $Q = t^{o(1)}$. Fix such an A, and write

$A = p + A'$, where $p \in \mathbb{Z}^d$ and $A' \in \mathcal{A}_Q^\square$. In particular, $E(A') \subset B(0, Q\sqrt{d})$. Since $x + \varphi E(A) = x + \frac{1}{n}p + \varphi E(A')$, we can assume by periodicity that $p \in \{0, \dots, n-1\}^d$.

Let $\delta \in (0, \frac{1}{3})$, take r, R as in (3.6), and choose $\tilde{n} = \tilde{n}(t) \in \mathbb{N}$ such that $1/\tilde{n} = \varphi^{1-3\delta+o(1)}$ and $1/\tilde{n} \geq 2R$. Let $\{\tilde{x}_1, \dots, \tilde{x}_{\tilde{n}^d}\}$ denote a grid of points in $\mathbb{T}^d$ with spacing $1/\tilde{n}$ (i.e., a translate of $G_{\tilde{n}}$), chosen in such a way that $d(x_0, \tilde{x}_j) > R$. To each grid point $\tilde{x}_j$, $j = 1, \dots, \tilde{n}^d$, associate in some deterministic way a point $x_j \in S$ with $d(x_j + \frac{1}{n}p, \tilde{x}_j) = d(x_j, \tilde{x}_j - \frac{1}{n}p) \leq h$ (this is always possible by the hypothesis on S). The choice of $\tilde{x}_j, x_j$ depends on t, but we suppress this dependence in our notation.

Since $h/\varphi = t^{o(1)}$, we have $r/h = \varphi^{-\delta+o(1)} \to \infty$. Since also $r/\varphi Q \to \infty$, we may take t large enough so that $h + \varphi Q\sqrt{d} < r < R < 1/\tilde{n}$, implying that $x_j + \varphi E(A) = x_j + \frac{1}{n}p + E(A') \subset B(\tilde{x}_j, r)$ for $j = 1, \dots, \tilde{n}^d$, and so we can apply Lemma 2.5 to the sets $x_j + \varphi E(A)$, uniformly in the choice of A and j.

Let $\sigma(s)$ be the total amount of time, up to time s, during which the Brownian motion is *not* making an excursion from $\partial B(\tilde{x}_j, r)$ to $\partial B(\tilde{x}_j, R)$ for any $j = 1, \dots, \tilde{n}^d$. In other words, $\sigma(s)$ is the Lebesgue measure of $[0,s] \setminus (\cup_{j=1}^{\tilde{n}^d} \cup_{i=1}^{\infty} [T_i'(\tilde{x}_j), T_i(\tilde{x}_j)])$. Define the stopping time $T'' = \inf\{s\colon\ \sigma(s) \geq t\}$. Clearly, $T'' \geq t$. Define N_j'' to be the number of excursions from $\partial B(\tilde{x}_j, r)$ to $\partial B(\tilde{x}_j, R)$ by time T'', and write $(\xi_i'(\tilde{x}_j), \xi_i(\tilde{x}_j))_{i=1,\dots,N_j''}$ for the starting and ending points of these excursions.

If $(x + \varphi E(A)) \cap W[0,t] \neq \varnothing$ for each $x \in S$, then necessarily, for each $j = 1, \dots, \tilde{n}^d$, at least one of the N_j'' excursions from $\partial B(\tilde{x}_j, r)$ to $\partial B(\tilde{x}_j, R)$ must hit $x_j + \varphi E(A)$. (Here we use that $d(x_0, \tilde{x}_j) > R$, which implies that the Brownian motion cannot hit $x_j + \varphi E(A)$ before the start of the first excursion.) These excursions are conditionally independent given $(\xi_i'(\tilde{x}_j), \xi_i(\tilde{x}_j))$ for $i = 1, \dots, N_j''$, $j = 1, \dots, \tilde{n}^d$. Applying Lemma 2.5 and (1.8), we get

$$
\begin{aligned}
&\mathbb{P}_{x_0}\big((x + \varphi E(A)) \cap W[0,t] \neq \varnothing \ \forall x \in S \,\big|\, (N_j'')_j, (\xi_i'(\tilde{x}_j), \xi_i(\tilde{x}_j))_{i,j}\big) \\
&\quad \leq \mathbb{P}_{x_0}\big((x_j + \varphi E(A)) \cap W[0,T''] \neq \varnothing \ \forall j \,\big|\, (N_j'')_j, (\xi_i'(\tilde{x}_j), \xi_i(\tilde{x}_j))_{i,j}\big) \\
&\quad = \prod_{j=1}^{\tilde{n}^d} \left(1 - \prod_{i=1}^{N_j''} \left(1 - \frac{\varphi^{d-2}\,\mathrm{Cap}\,E(A)}{\kappa_d\, r^{d-2}} (1 + o(1)) \right) \right) \\
&\quad \leq \exp\left[\sum_{j=1}^{\tilde{n}^d} \log\left(1 - (1 - (\varphi/r)^{d-2}(\kappa/\kappa_d + o(1)))^{N_j''} \right) \right].
\end{aligned} \tag{3.20}
$$

In this upper bound, which no longer depends on $(\xi_i'(\tilde{x}_j), \xi_i(\tilde{x}_j))_{i,j}$, the function $y \mapsto \log(1 - e^{cy})$ is concave, and hence we can replace each N_j'' by the empirical mean $\bar{N}'' = \tilde{n}^{-d} \sum_{j=1}^{\tilde{n}^d} N_j''$:

$$
\begin{aligned}
&\mathbb{P}_{x_0}\big((x + \varphi E(A)) \cap W[0,t] \neq \varnothing \ \forall x \in S \,\big|\, (N_j'')_j\big) \\
&\quad \leq \exp\left(\tilde{n}^d \log\left(1 - (1 - (\varphi/r)^{d-2}(\kappa/\kappa_d + o(1)))^{\bar{N}''} \right) \right) \\
&\quad \leq \exp\left[-\tilde{n}^d (1 - (\varphi/r)^{d-2}(\kappa/\kappa_d + o(1)))^{\bar{N}''} \right].
\end{aligned} \tag{3.21}
$$

Write $M = (1+\delta)N_d(t, r, R)$. On the event $\{\bar{N}'' \leq M\}$, the relations $(\varphi/r)^{d-2}M \sim$

$(1+\delta)d(d-2)^{-1}\log t$ and $\tilde{n}^d = t^{d/(d-2)-O(\delta)}$ imply that

$$\begin{aligned}
&\mathbb{1}_{\{\bar{N}'' \le M\}}\mathbb{P}_{x_0}\big((x+\varphi E(A)) \cap W[0,t] \neq \varnothing \; \forall x \in S \,\big|\, (N''_j)_j\big) \\
&\quad \le \exp\left[-t^{d/(d-2)-O(\delta)} \exp\left[-(\varphi/r)^{d-2} M(\kappa/\kappa_d + o(1))\right]\right] \\
&\quad = \exp\left[-t^{J_d(\kappa)-O(\delta)}\right].
\end{aligned} \tag{3.22}$$

Next, we will show that $\mathbb{P}_{x_0}(\bar{N}'' \ge M) \le \exp[-ct^{d/(d-2)-O(\delta)}]$. To that end, let $\pi^{(\tilde{n})}$ denote the projection map from the unit torus $\mathbb{T}^d$ to a torus of side length $1/\tilde{n}$. Under $\pi^{(\tilde{n})}$, every grid point $\tilde{x}_j$ maps to the same point $\pi^{(\tilde{n})}(\tilde{x}_j)$, and $\sigma(s)$ is the total amount of time the projected Brownian motion $\pi^{(\tilde{n})}(W)$ in $\pi^{(\tilde{n})}(\mathbb{T}^d)$ spends *not* making an excursion from $\partial B(\pi^{(\tilde{n})}(\tilde{x}_j), r)$ to $\partial B(\pi^{(\tilde{n})}(\tilde{x}_j), R)$, by time s. Moreover, $\tilde{n}^d \bar{N}'' = \sum_{j=1}^{\tilde{n}^d} N''_j$ can be interpreted as the number of such excursions in $\pi^{(\tilde{n})}(\mathbb{T}^d)$ completed by time T''.

Write $x \mapsto \tilde{n}x$ for the dilation that maps the torus $\pi^{(\tilde{n})}(\mathbb{T}^d)$ of side length $1/\tilde{n}$ to the unit torus $\mathbb{T}^d$. By Brownian scaling, $(\tilde{W}(u))_{u\ge 0} = (\tilde{n}\pi^{(\tilde{n})}(W(\tilde{n}^{-2}u)))_{u \ge 0}$ has the law of a Brownian motion in $\mathbb{T}^d$. Moreover, $\tilde{n}^d\bar{N}''$ can be interpreted as the number of excursions of $\tilde{W}(u)$ from $\partial B(\tilde{n}\pi^{(\tilde{n})}(\tilde{x}_j), \tilde{n}r)$ to $\partial B(\tilde{n}\pi^{(\tilde{n})}(\tilde{x}_j), \tilde{n}R)$ until the time spent not making such excursions first exceeds $\tilde{n}^2 t$, i.e., precisely the quantity $N'(\tilde{n}\pi^{(\tilde{n})}(\tilde{x}_j), \tilde{n}^2 t, \tilde{n}r, \tilde{n}R)$ from Section 2.1. We have $N_d(\tilde{n}^2 t, \tilde{n}r, \tilde{n}R) = \tilde{n}^d N_d(t,r,R)$, so Proposition 2.1 gives

$$\begin{aligned}
\mathbb{P}_{x_0}(\bar{N}'' \ge M) &= \mathbb{P}_{x_0}(\tilde{n}^d \bar{N}'' \ge \tilde{n}^d M) = \mathbb{P}_{\tilde{n}\pi^{(\tilde{n})}(x_0)}\big(N'(\tilde{n}\pi^{(\tilde{n})}(\tilde{x}_j), \tilde{n}^2 t, \tilde{n}r, \tilde{n}R) \ge \tilde{n}^d M\big) \\
&= \mathbb{P}_{\tilde{n}\pi^{(\tilde{n})}(x_0)}\big(N'(\tilde{n}\pi^{(\tilde{n})}(\tilde{x}_j), \tilde{n}^2 t, \tilde{n}r, \tilde{n}R) \ge (1+\delta) N_d(\tilde{n}^2 t, \tilde{n}r, \tilde{n}R)\big) \\
&\le \exp\left[-cN_d(\tilde{n}^2 t, \tilde{n}r, \tilde{n}R)\right] = \exp\left[-ct^{d/(d-2)-O(\delta)}\right].
\end{aligned} \tag{3.23}$$

Equations (3.22)–(3.23) imply that, for each fixed $A = p + A'$ with $\operatorname{Cap} E(A) \le \kappa$, we have

$$\mathbb{P}_{x_0}((x+\varphi E(A)) \cap W[0,t] \neq \varnothing \; \forall x \in S) \le \exp\left[-t^{d(1-\kappa/\kappa_d)/(d-2)-O(\delta)}\right]. \tag{3.24}$$

But the number of pairs (p, A') is at most $n^d \left|\mathcal{A}_Q^\square\right| = t^{d/(d-2)+o(1)} e^{O(Q)}$, by (3.2) and (3.16). Since $Q = t^{o(1)}$, a union bound completes the proof. $\square$

4 Proofs of Theorems 1.1–1.4

In proving Theorems 1.1–1.4, we bound hitting probabilities for Wiener sausages, e.g.

$$\mathbb{P}\left(\exists x \in \mathbb{T}^d \colon\, (x + \varphi_d(t)E) \cap W_{\rho(t)}[0,t] = \varnothing\right), \qquad E \subset \mathbb{R}^d, \tag{4.1}$$

in terms of the Brownian hitting probabilities estimated in Propositions 3.2 and 3.5–3.6, in which E is a rescaled lattice animal. In Section 4.1 we prove an approximation lemma for lattice animals, which leads directly to the proofs of Theorems 1.1–1.3 and Proposition 1.11. Proving Theorem 1.4 requires an additional argument to show that a component containing a given set is likely not to be much larger, and we prove this in Section 4.2. Finally in Section 4.3 we give the proof of Proposition 1.12.

4.1 Approximation by lattice animals

Lemma 4.1. *Let $\rho > 0$ and $n \in \mathbb{N}$ satisfy $\rho n \geq 2\sqrt{d}$, and let $\varphi > 0$. Then, given a bounded connected set $E \subset \mathbb{R}^d$, there is an $A \in \mathcal{A}^\square$ such that $E(A) = n^{-1}\varphi^{-1}A$ satisfies $E \subset E(A) \subset E_{\rho/\varphi}$ and, for any $x \in \mathbb{T}^d$, $0 \leq \tilde{\rho} \leq \frac{1}{4}\rho$,*

$$x + \varphi E \subset x' + \varphi E(A) \subset x + (\varphi E)_\rho \qquad \text{for some } x' \in G_n, \tag{4.2}$$

$$\big\{(x + \varphi E) \cap W_\rho[0,t] = \varnothing\big\} \subset \{\exists x' \in G_n\colon\ (x' + \varphi E(A)) \cap W[0,t] = \varnothing\}\,, \tag{4.3}$$

$$\big\{(x + \varphi E) \cap W_{\tilde{\rho}}[0,t] \neq \varnothing\big\} \subset \{(x + \varphi E(A)) \cap W[0,t] \neq \varnothing\}\,. \tag{4.4}$$

Proof. Let A be the union of all the closed unit cubes with centres in $\mathbb{Z}^d$ that intersect $n\varphi E_{\rho/4\varphi}$. This set is connected because E is connected, and therefore $A \in \mathcal{A}^\square$. Every cube in A is within distance $\sqrt{d}$ of some point of $n\varphi E_{\rho/4\varphi}$, so that $E \subset E_{\rho/4\varphi} \subset E(A) \subset E_{\rho/4\varphi+\sqrt{d}/n\varphi}$. By assumption, $\sqrt{d}/n \leq \rho/2$, so that $E(A) \subset E_{3\rho/4\varphi} \subset E_{\rho/\varphi}$ (see Figure 4(a)).

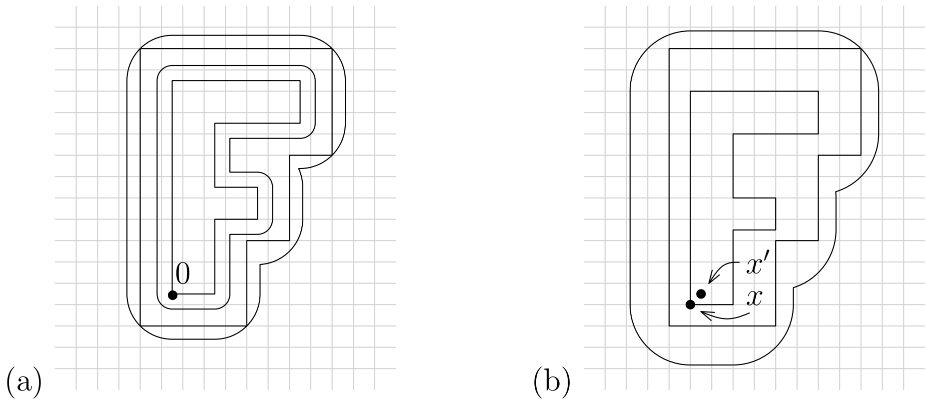

Figure 4: (a) From inside to outside: an F-shaped set E; the enlargement $E_{\rho/4\varphi}$; $E(A)$, the union of the rescaled cubes intersecting $E_{\rho/4\varphi}$; the bounding set $E_{3\rho/4\varphi}$. The grid shows the cubes in the definition of $E(A)$, rescaled to have side length $1/n\varphi$. The parameters ρ, n satisfy $\rho n = 2\sqrt{d}$. (b) From inside to outside (scaled by φ compared to part (a)): the prospective subset $x + \varphi E$ of $\mathbb{T}^d \setminus W_\rho[0,t]$; the approximating grid-aligned set $x' + \varphi E(A)$; the taboo set $x + (\varphi E)_\rho$ that the Brownian motion must not visit.

Given $x \in \mathbb{T}^d$, let $x' \in G_n$ satisfy $d(x, x') \leq \sqrt{d}/2n$. Then $x + \varphi E \subset x' + (\varphi E)_{\sqrt{d}/2n} \subset x' + \varphi E(A) \subset x + (\varphi E(A))_{\sqrt{d}/2n} \subset x + (\varphi E)_\rho$ since $\sqrt{d}/2n \leq \rho/4$ and $\varphi E(A) \subset (\varphi E)_{3\rho/4}$. See Figure 4(b). This proves (4.2); (4.3) follows immediately because $(x + \varphi E) \cap W_\rho[0,t] = \varnothing$ is equivalent to $(x + (\varphi E)_\rho) \cap W[0,t] = \varnothing$.

Similarly, since $(\varphi E)_{\rho/4} \subset \varphi E(A)$ and since $(x + \varphi E) \cap W_{\tilde{\rho}}[0,t] \neq \varnothing$ is equivalent to $(x + (\varphi E)_{\tilde{\rho}}) \cap W[0,t] \neq \varnothing$, the inclusion in (4.4) follows. $\square$

4.1.1 Proof of Theorem 1.3

Proof. Fix $E \subset \mathbb{R}^d$ compact with $\operatorname{Cap} E < \kappa_d$, and let $\delta > 0$ be arbitrary with $\operatorname{Cap} E + \delta < \kappa_d$. By Proposition 2.6(a), we can choose $r > 0$ so that $\operatorname{Cap}(E_r) \leq \operatorname{Cap} E + \frac{1}{2}\delta$. If E_r

is not already connected, then enlarge it to a connected set $E' \supset E_r$ by adjoining a finite number of line segments (this is possible because E_r is the r-enlargement of a compact set). Doing so does not change the capacity, so we may apply Proposition 2.6(a) again to find $r' > 0$ so that $\operatorname{Cap}((E')_{r'}) \leq \operatorname{Cap} E + \delta$.

Define $\rho_0(t) = r'\varphi_d(t)$ and $n(t) = \lceil 2\sqrt{d}/\rho_0(t) \rceil$, so that $\rho_0(t)n(t) \geq 2\sqrt{d}$ and the condition (3.16) from Proposition 3.6 holds. Since $\rho(t)/\varphi_d(t) \to 0$, we may choose t sufficiently large so that $\rho(t) \leq \frac{1}{4}\rho_0(t)$.

Apply Lemma 4.1 to E' with $\rho = \rho_0(t)$, $\tilde{\rho} = \rho(t)$, and $\varphi = \varphi_d(t)$. Note that if $(x + \varphi_d(t)E) \cap W_{\rho(t)}[0, t] \neq \varnothing$ for all $x \in S(t)$, then $(x + \varphi_d(t)E(A)) \cap W[0, t] \neq \varnothing$ for all $x \in S(t)$, where $\operatorname{Cap} E(A) \leq \operatorname{Cap}((E')_{\rho/\varphi}) = \operatorname{Cap}((E')_{r'}) \leq \operatorname{Cap} E + \delta$. By Proposition 3.6 with $\kappa = \operatorname{Cap} E + \delta$, this event has a probability that is at most $\exp[-t^{J_d(\operatorname{Cap} E) - O(\delta)}]$, and taking $\delta \downarrow 0$ we get the desired result. $\square$

4.1.2 Proof of Theorem 1.1

Proof. First consider $\kappa < \kappa_d$. Since $I_d(\kappa)$ is infinite for such κ, it suffices to show that $\lim_{t\to\infty} \log \mathbb{P}(\kappa^*(t, \rho(t)) \leq \kappa\varphi^{d-2})/\log t = -\infty$. Let $\kappa < \kappa' < \kappa_d$, and take E to be a ball of capacity κ'. If $\kappa^*(t, \rho(t)) \leq \kappa\varphi^{d-2}$, then no translate $x + \varphi_d(t)E$, $x \in \mathbb{T}^d$, can be a subset of $\mathbb{T}^d \setminus W_{\rho(t)}[0, t]$. Applying Theorem 1.3 with $S(t) = \mathbb{T}^d$, we conclude that $\mathbb{P}(\kappa^*(t, \rho(t)) \leq \kappa\varphi^{d-2}) \leq \exp[-t^{J_d(\kappa)+o(1)}]$, which implies the desired result.

Next consider the LDP upper bound for $\kappa \geq \kappa_d$. Since $\kappa \mapsto I(\kappa)$ is increasing and continuous on $[\kappa_d, \infty]$, it suffices to show that $\mathbb{P}(\kappa^*(t, \rho(t)) \geq \kappa\varphi^{d-2}) \leq t^{-I_d(\kappa)+o(1)}$ for $\kappa > \kappa_d$. Therefore, suppose that $x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W_{\rho(t)}[0, t]$ for some $x \in \mathbb{T}^d$ and $E \subset \mathbb{R}^d$ compact with $\operatorname{Cap} E \geq \kappa$. As in the proof of Theorem 1.3, define $n(t) = \lceil 2\sqrt{d}/\rho(t) \rceil$. Lemma 4.1 gives $(x' + \varphi_d(t)E(A)) \cap W[0, t] = \varnothing$ for some $x' \in G_{n(t)}$ and $\operatorname{Cap} E(A) \geq \operatorname{Cap} E \geq \kappa$. The condition in (1.16) on $\rho(t)$ implies the condition in (3.3) on $n(t)$, and therefore we may apply Proposition 3.2 to conclude that $\mathbb{P}(\kappa^*(t, \rho(t)) \geq \kappa\varphi^{d-2}) \leq t^{-I_d(\kappa)+o(1)}$.

Finally, the LDP lower bound for $\kappa \geq \kappa_d$ will follow (with E the ball of capacity κ, say) from the lower bound proved for Theorem 1.4 (see Section 4.2). $\square$

4.1.3 Proof of Theorem 1.2

Proof. As in the proof of Theorem 1.1, the lower bound will follow from the more specific lower bound proved for Theorem 1.4 (see Section 4.2).

Choose $n(t)$ such that $n(t) \geq 2\sqrt{d}/\rho(t)$ and the hypotheses of Proposition 3.5 hold. (The conditions on $n(t)$ are mutually consistent because $2\sqrt{d}/\rho(t) = O(1/\varphi_d(t))$.) Given any component C containing a ball of radius $\rho(t)$ and having the form $C = x + \varphi_d(t)E$ for $\operatorname{Cap} E \geq \kappa$, apply Lemma 4.1 to find $x'_C \in G_{n(t)}$ and $A_C \in \mathcal{A}^\square$ such that $C \subset x'_C + \varphi_d(t)E(A_C) \subset C_{\rho(t)} \subset \mathbb{T}^d \setminus W[0, t]$. The pairs $(x'_C, E(A_C))$ so constructed must be distinct: for $C' \neq C$, we have $x'_{C'} + \varphi_d(t)E(A_{C'}) \subset C'_{\rho(t)} \subset (\mathbb{T}^d \setminus C)_{\rho(t)} = \mathbb{T}^d \setminus C_{-\rho(t)}$, and since $C_{-\rho(t)}$ is non-empty by assumption, it follows that $C \not\subseteq x'_{C'} + \varphi_d(t)E(A_{C'})$. We therefore conclude that $\chi_{\rho(t)}(t, \kappa) \leq \chi_+^\square(t, n(t), \kappa)$, so the required upper bound follows from Proposition 3.5. $\square$

4.1.4 Proof of Proposition 1.11

Proof. Abbreviate $\varphi = \varphi_d(t), \rho = \rho(t)$. It suffices to bound the probability that $\mathbb{T}^d \setminus W_\rho[0,t]$ has a component of diameter at least $\frac{1}{2}$, since the mapping $x + y \mapsto y$ from $B(x,r) \subset \mathbb{T}^d$ to $B(0,r) \subset \mathbb{R}^d$ is a well-defined local isometry if $r < \frac{1}{2}$.

Suppose that $x \in \mathbb{T}^d \setminus W_\rho[0,t]$ belongs to a connected component intersecting $\partial B(x, \frac{1}{2})$. Then there is a bounded connected set $E \subset \mathbb{R}^d$ such that $(x + \varphi E) \cap W_\rho[0,t]$ and $E \cap \partial B(0, \frac{1}{2}\varphi^{-1}) \neq \varnothing$ (see Figure 5). Define $n = n(t) = \lceil 2\sqrt{d}/\rho \rceil$ and apply Lemma 4.1 to conclude that $(x' + \varphi E(A)) \cap W[0,t] = \varnothing$ with $E \subset E(A)$, $A \in \mathcal{A}^\square$, $x' \in G_n$. Since $E(A)$ contains E, it has diameter at least $\frac{1}{2}\varphi^{-1}$, so A has diameter at least $\frac{1}{2}n$ and must consist of at least $n/(2\sqrt{d})$ unit cubes. Since $\rho = o(\varphi)$ and $\varphi = t^{-d/(d-2)+o(1)}$, we have $n \geq t^{d/(d-2)+o(1)}$. The hypothesis in (1.16) implies that $n\varphi = o((\log t)^{1/d})$, as in condition (3.3) from Proposition 3.2. Therefore $\operatorname{Vol} E(A) \geq (n\varphi)^{-d} n/(2\sqrt{d}) \geq t^{d/(d-2)+o(1)}$, and in particular $\operatorname{Vol} E(A) \to \infty$. By (1.44), $\operatorname{Cap} E(A) \to \infty$ also. Thus, if $\mathbb{T}^d \setminus W_\rho[0,t]$ has a component of diameter at least $\frac{1}{2}$, then the event in Proposition 3.2 occurs, with κ arbitrarily large for $t \to \infty$. By Proposition 3.2, the probability of this occuring is negligible, as claimed. $\square$

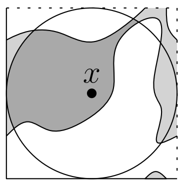

Figure 5: A large connected component of $\mathbb{T}^d \setminus W_\rho[0,t]$ that is not isometric to a subset of $\mathbb{R}^d$ (shading) and a possible choice of the set $x + \varphi E$ (dark shading).

This proof is unchanged if the radius $\frac{1}{2}$ is replaced by any $\delta \in (0, \frac{1}{2})$, which shows that the maximal diameter $D(t, \rho(t))$ satisfies $D(t, \rho(t)) \to 0$ in $\mathbb{P}$-probability when (1.16) holds (see Section 1.6.5).

4.2 Proof of Theorem 1.4

In Theorems 1.1–1.3 we deal with components that contain a subset $x + \varphi_d(t)E$ of a given form. Theorem 1.4 adds the requirement that the component containing such a subset should not extend further than distance $\delta\varphi_d(t)$ from $x + \varphi_d(t)E$. In the proof, we will bound the probability that the component extends no further than distance $\rho(t)$ from $x + \varphi_d(t)E$, but only for sets $E \in \mathcal{E}_c^\square$ of the following kind: define

$$\mathcal{E}_c^\square = \left\{ E \in \mathcal{E}_c \colon E = \tfrac{1}{n} A \text{ for some } A \in \mathcal{A}^\square \right\} \tag{4.5}$$

to be the collection of sets in $\mathcal{E}_c$ that are rescalings of lattice animals.

Note that, unlike in Section 3, the scaling factor $\frac{1}{n}$ in (4.5) is fixed and does not depend on t. We begin by showing that the collection $\mathcal{E}_c^\square$ is dense in $\mathcal{E}_c$.

Lemma 4.2. *Given $E \in \mathcal{E}_c$ and $\delta > 0$, there exists $E^\square \in \mathcal{E}_c^\square$ with $E \subset E^\square \subset E_\delta$.*

Lemma 4.2 will allow us to prove Theorem 1.4 only for $E \in \mathcal{E}_c^\square$.

Proof. For $y \notin E$, define

$$b(y) = \sup \left\{ r > 0\colon\ y \text{ belongs to the unbounded component of } \mathbb{R}^d \setminus E_r \right\}. \tag{4.6}$$

Since $\mathbb{R}^d \setminus E$ is open and connected, $b(y)$ is continuous and positive on $\mathbb{R}^d \setminus E$. By compactness, we may choose $\eta \in (0, \delta)$ such that $b(y) > \eta$ for $y \notin E_\delta$. Apply Lemma 4.1 (with ρ and φ replaced by η and 1, and n sufficiently large) to find $E' = \frac{1}{n}A$ with $E \subset E' \subset E_\eta$. The set E' is a rescaled lattice animal, but $\mathbb{R}^d \setminus E'$ might not be connected. However, if y belongs to a bounded component of $\mathbb{R}^d \setminus E'$, then $b(y) \leq \eta$ by construction: since $E' \subset E_\eta$, y cannot belong to the unbounded component of $\mathbb{R}^d \setminus E_\eta$. By choice of η, it follows that every bounded component of $\mathbb{R}^d \setminus E'$ is contained in E_δ. Thus, if we define $E^\square$ to be E' together with these bounded components (see Figure 6), then $E^\square \in \mathcal{E}_c^\square$ and $E^\square \subset E_\delta$, as claimed. $\square$

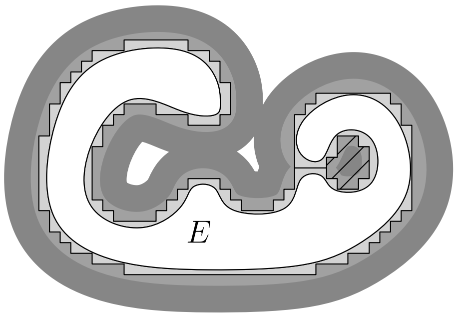

Figure 6: A set E (white) and its enlargement E_δ (dark shading). Every bounded component of $\mathbb{R}^d \setminus E_\delta$ can reach infinity without touching E_η (medium shading). A set E' (light shading) with $E \subset E' \subset E_\eta$ may disconnect a region from infinity (diagonal lines), but this region must belong to E_δ.

In the proof of Theorem 1.4, we adapt the concept of (N, φ, r, R)-successful from Definition 2.3 to formulate the desired event in terms of excursions. To this end we next introduce the sets and events that we will use. In the remainder of this section, we abbreviate $\varphi = \varphi_d(t)$, $\rho = \rho(t)$, $I_d(\kappa) = I(\kappa)$ and $J_d(\kappa) = J(\kappa)$.

Fix $E \in \mathcal{E}_c^\square$ and $\delta > 0$. We may assume that $E \subset B(0, a)$ with $a > \delta$. Let $\eta \in (0, \frac{1}{2})$ be small enough that $\kappa_d \eta^{d-2} < \operatorname{Cap} E$. Set $r = \varphi^{1-\eta}$, $R = \varphi^{1-2\eta}$, and let $\{x_0, \ldots, x_k\} \subset \mathbb{T}^d$ denote a maximal collection of points in $\mathbb{T}^d$ satisfying $d(x_0, x_j) > R$ and $d(x_j, x_k) > 2R$ for $j \neq k$, so that

$$K = R^{-d-o(1)} = t^{d/(d-2)+O(\eta)} \tag{4.7}$$

Take t large enough that $\rho < \frac{1}{2}\delta\varphi$ and $R < \frac{1}{2}$. Set $N = (1+\eta)N_d(t, r, R)$ (see (2.5)).

Choose $q = q(t)$ with $q > 2a + \delta$, $q \geq \log t$, and $q = (\log t)^{O(1)}$. Let $\{y_1, \ldots, y_L\} \subset B(0, 2q) \setminus E_\delta$ denote a maximal collection of points in $B(0, 2q) \setminus E_\delta$ satisfying $d(y_\ell, E) \geq \delta$, $d(y_\ell, y_m) \geq \frac{1}{2}\rho/\varphi$ for $\ell \neq m$, so that $L = O((q\varphi/\rho)^d) = (\log t)^{O(1)}$ by (1.16).

Let $Z = \partial(E_{\rho/\varphi}) \cup (\cup_{z \in B(0,2a)\cap\eta\mathbb{Z}^d} \partial B(z,\eta) \setminus E_{\rho/\varphi})$ (see Figure 7: Z consists of a $(d-1)$-dimensional shell around E together with a finite number of $(d-1)$-dimensional spheres). Let $\{z_1, \dots, z_M\} \subset Z$ denote a maximal collection of points in Z with $d(z_m, z_p) \geq \frac{1}{2}\rho/\varphi$ for $m \neq p$. Since Z is $(d-1)$-dimensional, we have $M = O((\rho/\varphi)^{d-1})$.

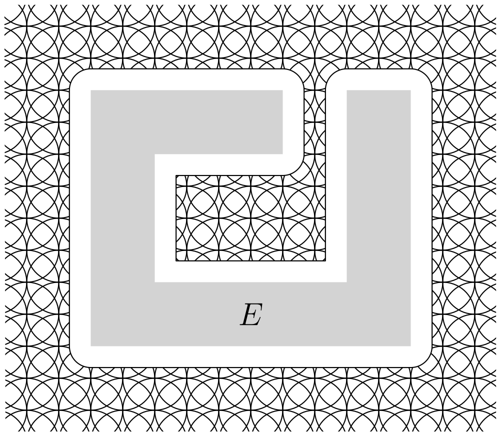

Figure 7: The set E (shaded) and part of the $(d-1)$-dimensional set Z.

For $j = 1, \dots, K$, define the following events.

- $F_1(j) = \left\{\frac{1}{2}N \leq N(x_j, t, r, R) \leq N\right\}$ is the event that W makes between $\frac{1}{2}N$ and N excursions from $\partial B(x_j, r)$ to $\partial B(x_j, R)$ by time t.
- $F_2(j)$ is the event that $(x_j, E_{\rho/\varphi})$ is $(\lfloor N \rfloor, \varphi, r, R)$-successful.
- $F_3(j)$ is the event that, for each $\ell = 1, \dots, L$, the i^{th} excursion from $\partial B(x_j, r)$ to $\partial B(x_j, R)$ hits $x_j + B(\varphi y_\ell, \frac{1}{2}\rho)$ for some $i = i(\ell) \in \{1, \dots, \lfloor N/4 \rfloor\}$.
- $F_4(j)$ is the event that, for each $m = 1, \dots, M$, the i^{th} excursion from $\partial B(x_j, r)$ to $\partial B(x_j, R)$ hits $x_j + B(\varphi z_m, \frac{1}{2}\rho)$ for some $i = i(m) \in \{\lfloor N/4 \rfloor + 1, \dots, \lfloor N/2 \rfloor\}$.
- $F_5(j)$ is the event that $\mathbb{T}^d \setminus W_\rho[0,t]$ contains no component of capacity at least $\varphi^{d-2} \operatorname{Cap} E$ disjoint from $B(x_j, 2q\varphi)$.
- $F(j) = F_1(j) \cap F_2(j) \cap F_3(j)$.
- $F_{\max}(j) = F_1(j) \cap F_2(j) \cap F_3(j) \cap F_4(j) \cap F_5(j)$.

Lemma 4.3. *On $F(j)$, the component of $\mathbb{T}^d \setminus W_\rho[0,t]$ containing $x_j + \varphi E$ satisfies condition $(\mathcal{C}(t, \rho, E, E'))$ with $E' = E_\delta$. Furthermore, $F_{\max}(j) \subset F_\rho(t, E, E_\delta)$ for t sufficiently large.*

Proof. Note that if $F_1(j) \cap F_2(j)$ occurs, then $x_j + \varphi E \subset \mathbb{T}^d \setminus W_\rho[0,t]$. If $F_1(j) \cap F_3(j)$ occurs, then the set $x_j + \cup_{\ell=1}^{L} B(\varphi y_\ell, \frac{1}{2}\rho)$ is entirely covered by the Wiener sausage. By choice of $\{y_1, \ldots, y_L\}$, this set contains $x_j + (B(0, 2q\varphi) \setminus \varphi E_\delta)$, and consequently $\big(\mathbb{T}^d \setminus W_\rho[0,t]\big) \cap B(x_j, 2q\varphi) \subset x_j + \varphi E_\delta$.

We have therefore shown that, on $F(j)$, $\mathbb{T}^d \setminus W_\rho[0,t]$ has a component containing $x_j + \varphi E$ and satisfying condition $\mathcal{C}(t, \rho, E, E_\delta)$. To show further that $F_{\max}(j) \subset F_\rho(t, E, E_\delta)$, we will show any other component must have capacity smaller than $\varphi^{d-2} \operatorname{Cap} E$.

If $F_1(j) \cap F_4(j)$ occurs, then $x_j + \varphi Z$ is entirely covered by the Wiener sausage, by choice of $\{z_1, \ldots, z_M\}$. By choice of Z, all components of $B(x_j, a\varphi) \setminus (x_j + \varphi Z)$, other than any components that are subsets of $x_j + \varphi E_{\rho/\varphi} = x_j + (\varphi E)_\rho$, must be contained in a ball of radius $\eta\varphi$, and in particular have capacity at most $\kappa_d (\eta\varphi)^{d-2} < \varphi^{d-2} \operatorname{Cap} E$.

Finally, if $F_5(j)$ occurs, then the component of largest capacity cannot occur outside $B(x_j, 2q\varphi)$, and therefore must be the component of largest capacity contained in $x_j + (\varphi E)_\rho$.

It therefore remains to show that the component of largest capacity in $x_j + (\varphi E)_\rho$ is in fact the component containing $x_j + \varphi E$. Suppose that $a \in x_j + \varphi E$ is the centre of a $(d-1)$-dimensional ball of radius ρ that is completely contained in some face of $x_j + \varphi E$, and let b be a point at distance at most ρ from a along the line perpendicular to the face (see Figure 8). If both $x_j + \varphi E$ and b are contained in $\mathbb{T}^d \setminus W_\rho[0,t]$, then so is the line segment from a to b, so that b belongs to the same component as $x_j + \varphi E$.

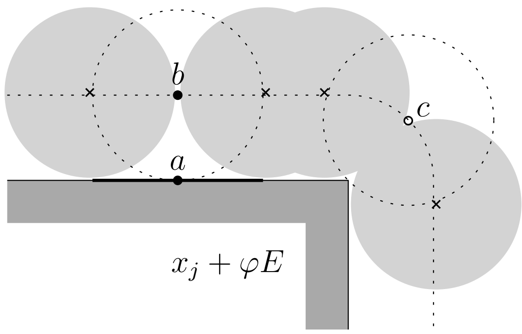

Figure 8: A point b near the centre a of a ball (thicker line) on a face of $x_j + \varphi E$, and a point c near the boundary of a face. The Brownian path must not touch the dotted lines, but the Wiener sausage can fill the shaded circles by visiting the crossed points. The point c can belong to a different component than $x_j + \varphi E$, but b cannot.

We therefore conclude that, on $F_{\max}(j)$, any point of $x_j + (\varphi E)_\rho$ that is not in the same component as $x_j + \varphi E$ must lie within distance 2ρ of the boundary of some face of $x_j + \varphi E$. Write H for the set of boundaries of faces of E. Since H is $(d-2)$-dimensional, its capacity is 0, and therefore $\operatorname{Cap}((\varphi H)_{2\rho}) = \varphi^{d-2} \operatorname{Cap}(H_{2\rho/\varphi}) = o(\varphi^{d-2})$ by Proposition 2.6(a), since $\rho/\varphi \to 0$. In particular, for t sufficiently large the component of largest capacity in $x_j + (\varphi E)_\rho$ must be the component containing $x_j + \varphi E$, which completes the proof of Lemma 4.3. $\square$

Proof of Theorem 1.4. Because of the upper bound proved for Theorems 1.1–1.2, we

need only prove the lower bounds

$$\mathbb{P}\left(F_\rho(t,E,E_\delta)\right) \geq t^{-I(\operatorname{Cap}E)-o(1)}, \qquad \operatorname{Cap}E \geq \kappa_d, \delta > 0. \tag{4.8}$$

and

$$\chi_\rho(t,E,E_\delta) \geq t^{J(\operatorname{Cap}E)-o(1)} \text{ with high probability,} \qquad \operatorname{Cap}E < \kappa_d, \delta > 0. \tag{4.9}$$

Moreover, it suffices to prove (4.8)–(4.9) under the assumption that $E \in \mathcal{E}_c^\square$ and, in (4.8), that $\operatorname{Cap}E > \kappa_d$. Indeed, given any $\delta' \in (0, \frac{1}{2}\delta)$, apply Lemma 4.2 to find $E^\square \in \mathcal{E}_c^\square$ with $E \subset E^\square \subset E_{\delta'}$. By adjoining, if necessary, a sufficiently small cube to $E^\square$, we may assume that $\operatorname{Cap}E^\square > \operatorname{Cap}E$. Apply (4.8)–(4.9) with E and δ replaced by $E^\square$ and δ', respectively. Proposition 2.6(a) implies that $\operatorname{Cap}E^\square \downarrow \operatorname{Cap}E$ as $\delta' \downarrow 0$. Since $\kappa \mapsto J(\kappa)$ is continuous, we conclude that the bounds for $E \in \mathcal{E}_c$ follows from those for $E \in \mathcal{E}_c^\square$.

We next relate the left-hand side of (4.8) to the events $F_1(j), \dots, F_5(j)$. Noting that $F_1(j) \cap F_2(j) \cap F_1(k) \cap F_2(k) \subset F_5(j)^c$ for $j \neq k$, Lemma 4.3 implies that

$$\begin{aligned}
&\mathbb{P}\left(F_\rho(t,E,E_\delta)\right) \geq \sum_{j=1}^K \mathbb{P}_{x_0}(F(j)) \\
&\geq \sum_{j=1}^K \mathbb{P}_{x_0}(F_2(j) \cap F_3(j) \cap F_4(j)) - \sum_{j=1}^K \mathbb{P}_{x_0}(F_1(j)^c) - \sum_{j=1}^K \mathbb{P}_{x_0}(F_1(j) \cap F_2(j) \cap F_5(j)^c).
\end{aligned} \tag{4.10}$$

We will bound each of the sums in the right-hand side of (4.10).

Applying Proposition 2.1 and (4.7) (and noting that $N_d(t,r,R) = t^{\eta+o(1)}$ and that $\frac{1}{2}N/N_d(t,r,R) = \frac{1}{2}(1+\eta) < \frac{3}{4}$), we see that the second sum in the right-hand side of (4.10) is at most $t^{d/(d-2)+O(\eta)}\exp[-ct^{\eta+o(1)}]$. This term will be negligible compared to the scale of (4.8).

For the last sum in (4.10), we assume that $\operatorname{Cap}E > \kappa_d$ and use Lemma 3.4. Set $h(t) = 2q\varphi$, and note that $h(t)/(\varphi \log t) \geq 1$ by assumption on q. If $F_1(j) \cap F_2(j) \cap F_5(j)^c$ occurs, then, by Lemma 4.1, there are lattice animals $A, A' \in \mathcal{A}^\square$ with $\operatorname{Cap}E(A), \operatorname{Cap}E(A') \geq \operatorname{Cap}E$ and a point $x' \in \mathbb{T}^d \setminus B(x_j, 2q\varphi)$ with $(x_j + \varphi E(A)) \cap W[0,t] = (x' + \varphi E(A')) \cap W[0,t] = \varnothing$. By Lemma 3.4 with $\kappa^{(1)} = \kappa^{(2)} = \operatorname{Cap}E$, we have

$$\mathbb{P}_{x_0}(F_1(j) \cap F_2(j) \cap F_5(j)^c) \leq t^{-d\operatorname{Cap}(E)/[(d-2)\kappa_d] - I(\operatorname{Cap}E)+o(1)}. \tag{4.11}$$

Hence the last sum in (4.10) is at most $t^{-2I(\operatorname{Cap}E)+O(\eta)}$. Since $I(\operatorname{Cap}E) > 0$, this term is also negligible, for η sufficiently small, compared to the scale of (4.8). (This is the only part of the proof where $\operatorname{Cap}E > \kappa_d$ is used.)

We have therefore proved that (4.8) will follow if we can give a suitable lower bound for the first sum on the right-hand side of (4.10). Using again the asymptotics (4.7) for K, (4.8) will follow from

$$\mathbb{P}_{x_0}\left(F_2(j) \cap F_3(j) \cap F_4(j) \,\middle|\, ((\xi_i'(x_j), \xi_i(x_j))_{i=1}^N)_{j=1}^K\right) \geq t^{-d\operatorname{Cap}E/[(d-2)\kappa_d] - O(\eta)}. \tag{4.12}$$

In fact, (4.12) also implies (4.9). On the event $\cap_{j=1}^K F_1(j)$ (which occurs with high probability, by Proposition 2.1), Lemma 4.3 implies that $\chi_\rho(t, E, E_\delta)$ is at least as large as the number of $j \in \{1, \dots, K\}$ for which $F_2(j) \cap F_3(j)$ occurs. Since the events $F_2(j) \cap F_3(j)$ are conditionally independent for different j given the starting and ending points $((\xi_i'(x_j), \xi_i(x_j))_{i=1}^N)_{j=1}^K$, (4.12) and (4.7) immediately imply that $\chi_\rho(t, E, E_\delta) \geq t^{J_d(\kappa)-O(\eta)}$ with high probability (cf. the proof of Proposition 3.5 in Section 3.2.1).

It therefore remains to prove (4.12). To do so, we will condition on not hitting $x_j + (\varphi E)_\rho$ and use the following lemma to estimate the conditional probability of hitting small nearby balls. Note that, conditional on the occurence of $F_2(j)$ and the starting and ending points $(\xi_i'(x_j), \xi_i(x_j))_{i=1}^N$, the events $F_3(j)$ and $F_4(j)$ are independent.

Lemma 4.4. *Fix $E \in \mathcal{E}_c^\square$ and $\delta > 0$, and let $0 < \rho < \varphi < r < R < \frac{1}{2}$. Then there is an $\epsilon > 0$ such that if $\rho/\varphi < \epsilon$, $\varphi/r < \epsilon$ and $r/R \leq \frac{1}{2}$, then, uniformly in $x \in \mathbb{T}^d$, $\xi' \in \partial B(x, r)$, and $\xi \in \partial B(x, R)$,*

$$\begin{aligned} &\mathbb{P}_{\xi',\xi}\big((x + B(\varphi y, \tfrac{1}{2}\rho)) \cap W[0, \zeta_R] \neq \varnothing \,\big|\, (x + (\varphi E)_\rho) \cap W[0, \zeta_R] = \varnothing\big) \\ &\qquad\qquad \geq \begin{cases} \epsilon(\varphi/r)^{d-2}(\rho/\varphi)^{d-2}, & \text{if } y \in B(0, r/\varphi) \setminus E_\delta, \\ \epsilon(\varphi/r)^{d-2}(\rho/\varphi)^{\alpha}, & \text{if } y \in E_\delta \setminus E_{\rho/\varphi}, \end{cases} \end{aligned} \tag{4.13}$$

where $\alpha > d - 2$ is some constant depending only on d.

We give the proof of Lemma 4.4 in Section A.2.

The event $F_3(j)$ says that all $(x_j, B(y_\ell, \frac{1}{2}\rho/\varphi))$, $\ell = 1, \dots, L$, are *not* $(\lfloor N/4 \rfloor, \varphi, r, R)$-successful. Lemma 4.4 implies (as in the proof of Proposition 2.4) that, uniformly in ℓ,

$$\begin{aligned} &\mathbb{P}_{x_0}\big((x_j, B(y_\ell, \tfrac{1}{2}\rho/\varphi)) \text{ is } (\lfloor N/4 \rfloor, \varphi, r, R)\text{-successful} \,\big|\, F_2(j)\big) \\ &\quad \leq \big(1 - \epsilon(\rho/r)^{d-2}(\rho/\varphi)\big)^{\lfloor N/4 \rfloor} = \exp\left[-\epsilon \lfloor N/4 \rfloor (\varphi/r)^{d-2}(\rho/\varphi)^{d-1}(1 + o(1))\right]. \end{aligned} \tag{4.14}$$

Recalling (1.3) and (2.5), we have $N(\varphi/r)^{d-2} \geq (d/(d-2) + O(\eta)) \log t$, so that

$$\begin{aligned} &\mathbb{P}_{x_0}\big(\text{some } (x_j, B(y_\ell, \tfrac{1}{2}\rho/\varphi)) \text{ is } (\lfloor N/4 \rfloor, \varphi, r, R)\text{-successful} \,\big|\, F_2(j)\big) \\ &\quad \leq L \exp\left[-\epsilon\left(\tfrac{1}{4}d/(d-2) + O(\eta)\right)\left(\frac{(\log t)^{1/d}\rho}{\varphi}\right)^{d-2} (\log t)^{2/d}\right]. \end{aligned} \tag{4.15}$$

By (1.16), $(\log t)^{1/d}\rho/\varphi \to \infty$, whereas $L = (\log t)^{O(1)}$. Hence, the conditional probability in (4.15) is $o(1)$ and $\mathbb{P}\left(F_3(j) \,|\, F_2(j)\right) = 1 - o(1)$.

For $F_4(j)$, write $k = \lfloor N/2 \rfloor - \lfloor N/4 \rfloor$ and $p = \epsilon(\varphi/r)^{d-2}(\rho/\varphi)^\alpha$. Lemma 4.4 states that, conditional on $F_2(j)$, each ball $x_j + B(\varphi z_m, \frac{1}{2}\rho)$ has a probability at least p of being hit during each of the k excursions from $\partial B(x_j, r)$ to $\partial B(x_j, R)$ in the definition of $F_4(j)$. It follows that $\mathbb{P}\left(F_4(j) \,|\, F_2(j)\right)$ is at least the probability that a Binomial(k, p) random variable has value M or larger. We have $p \to 0$ and $k - M \to \infty$ as $t \to \infty$, so

using Stirling's approximation, we get

$$\mathbb{P}_{x_0}(F_4(j)\,|\,F_2(j)) \geq \binom{k}{M} p^M (1-p)^{k-M} = \frac{k^k p^M (1-p)^{k-M}}{M^M (k-M)^{k-M}(\sqrt{2\pi}+o(1))}$$
$$\geq \left(\frac{kp}{M}\right)^M \frac{(1-p)^k}{\sqrt{2\pi}+o(1)} = \exp\left[-M\log(M/kp) - O(kp) - O(1)\right]. \quad (4.16)$$

Observe that $kp = e^{O(1)} N_d(t,r,R)(\varphi/r)^{d-2}(\rho/\varphi)^\alpha = e^{O(1)}(\rho/\varphi)^\alpha \log t$. The assumption $\rho/\varphi \to 0$ implies that $kp = o(\log t)$. On the other hand, recall that $M = O((\varphi/\rho)^{d-1})$, so that $M/kp = e^{O(1)}(\varphi/\rho)^{\alpha+d-1}/\log t$. The hypothesis (1.16) means that $\varphi/\rho = o((\log t))^{1/d}$. Consequently, $M = o((\log t)^{(d-1)/d})$ and $\log(M/kp) \leq O(\log\log t)$. In particular, $M\log(M/kp) \leq o(\log t)$, and we conclude that

$$\mathbb{P}_{x_0}(F_4(j)\,|\,F_2(j)) = \exp\left(-o(\log t)\right) = t^{o(1)}. \quad (4.17)$$

Combining (4.15), (4.17), and Proposition 2.4, we obtain

$$\mathbb{P}_{x_0}(F_2(j)\cap F_3(j)\cap F_4(j)) = \mathbb{P}_{x_0}(F_2(j))\mathbb{P}_{x_0}\left(F_3(j)\,|\,F_2(j)\right)\mathbb{P}_{x_0}\left(F_4(j)\,|\,F_2(j)\right)$$
$$= t^{-d\,\mathrm{Cap}(E_{\rho/\varphi})/[(d-2)\kappa_d]+O(\eta)}\left[1-o(1)\right] t^{o(1)} = t^{-d\,\mathrm{Cap}(E)/[(d-2)\kappa_d]+O(\eta)}. \quad (4.18)$$

We have therefore verified (4.12), and this completes the proof. □

4.3 Proof of Proposition 1.12

Proof. $\mathbb{T}^d \setminus W[0,t]$ is open since $W[0,t]$ is the (almost surely) continuous image of a compact set.

Consider first a Brownian motion $\tilde{W}$ in $\mathbb{R}^d$. Define

$$\tilde{Z} = \bigcup_{q,q'\in\mathbb{Q}} \bigcup_{1\leq i<j\leq d} \left\{(x_1,\ldots,x_d)\in\mathbb{R}^d : x_i = q, x_j = q'\right\} \quad (4.19)$$

and note that $\tilde{Z}$ is the inverse image $\pi_0^{-1}(Z)$ of a path-connected, locally path-connected, dense subset $Z = \pi_0(\tilde{Z}) \subset \mathbb{T}^d$ (where $\pi_0\colon ,\mathbb{R}^d \to \mathbb{T}^d$ is the canonical projection map). Since $\tilde{Z}$ is the countable union of $(d-2)$-dimensional subspaces, $\tilde{W}\,[0,\infty)$ does not intersect $\tilde{Z}$, except perhaps at the starting point, with probability 1. Projecting onto $\mathbb{T}^d$, it follows that $W\,[0,\infty)$ intersects Z in at most one point, and in particular $\mathbb{T}^d \setminus W\,[0,\infty)$ contains a path-connected, locally path-connected, dense subset. This implies the remaining statements in Proposition 1.12. □

5 Proofs of Corollaries 1.6–1.10

5.1 Proof of Corollary 1.6

Proof. (1.25) follows immediately from the more precise statements in (1.26)–(1.27). By monotonicity and continuity, it suffices to show (1.26) for $\mathrm{Cap}\,E > \kappa_d$.

Consider first the lower bounds in (1.26)–(1.27). Replace E by the compact set $\operatorname{clo}(E)$ (by hypothesis, this does not change the value of $\operatorname{Cap} E$). Let $\kappa > \operatorname{Cap} E$ be arbitrary and use Proposition 2.6(a) to find $r > 0$ such that $\operatorname{Cap}(E_r) \leq \kappa$. Adjoin finitely many lines to E_r to make it into a connected set E' (as in the proof of Theorem 1.3) and then adjoin any bounded components of $\mathbb{R}^d \setminus E'$ to form a set $E'' \in \mathcal{E}_c$ that satisfies the conditions of Theorem 1.4. For $\operatorname{Cap} E \geq \kappa_d$, Theorem 1.4 implies that $x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]$ for some $x \in \mathbb{T}^d$, with probability at least $t^{J_d(\kappa)-o(1)}$. If instead $\operatorname{Cap} E < \kappa_d$, then it is no loss of generality to assume that $\kappa < \kappa_d$ also. Then Theorem 1.4 shows that there are at least $t^{J_d(\kappa)-o(1)}$ components containing translates $x+\varphi_d(t)E$; these translates are necessarily disjoint. In both cases we conclude by taking $\kappa \downarrow \operatorname{Cap} E$.

For the upper bounds, we will shrink the set E. The results nearly follow from Theorems 1.1–1.2, since the existence of $x+\varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]$ implies the existence of $x + (\varphi_d(t)E)_{-\rho(t)} \subset \mathbb{T}^d \setminus W_{\rho(t)}[0,t]$. However, the set E might not be connected. To handle this possibility, we will appeal directly to Lemmas 3.3 and 3.7.

Let $\kappa \in (\kappa_d, \operatorname{Cap} E)$ (for (1.26)) or $\kappa \in (0, \operatorname{Cap} E)$ (for (1.27)) be arbitrary. Apply Proposition 2.6(c) to find an $r > 0$ such that $\operatorname{Cap}(E_{-2r}) > \kappa$. The enlargement $(E_{-2r})_r$ has a finite number k of components, by boundedness. Set $\rho = \rho(t) = r\varphi_d(t)$ and choose $n = n(t)$ such that $n(t) \geq 2\sqrt{d}/\rho(t)$ and the hypotheses of Proposition 3.5 hold. (As in the proof of Theorem 1.2, these conditions on $n(t)$ are mutually consistent.) Apply Lemma 4.1 to each of the k components of $(E_{-2r})_r$ to obtain a set $E^\square = \cup_{j=1}^k E(A^{(j)})$ satisfying $(E_{-2r})_r \subset E^\square \subset (E_{-2r})_{2r} \subset E$. Thus, $\operatorname{Cap} E^\square \geq \kappa$. Furthermore, given $x \in \mathbb{T}^d$ there is $x' \in G_{n(t)}$ such that $x' + \varphi_d(t)E^\square \subset x + \varphi_d(t)((E_{-2r})_{2r}) \subset x + \varphi_d(t)E$. Define $h(t) = C\varphi_d(t)$, where C is a constant large enough so that $E \subset B(0,C)$. For $\operatorname{Cap} E > \kappa_d$, we can then apply Lemma 3.3 to conclude that $\mathbb{P}(\exists x \in \mathbb{T}^d \colon x + \varphi_d(t)E \subset \mathbb{T}^d \setminus W[0,t]) \leq t^{J_d(\kappa)+o(1)}$. For $\operatorname{Cap} E < \kappa_d$, Lemma 3.7 implies that $\chi(t,E) \leq \chi_+^\square(t, n(t), \kappa, h(t)) \leq t^{J_d(\kappa)+o(1)}$ with high probability. In both cases take $\kappa \uparrow \operatorname{Cap} E$. $\square$

5.2 Proof of Corollaries 1.7–1.8

Proof. Note the scaling relation

$$\lambda(\varphi D) = \varphi^{-2}\lambda(D). \tag{5.1}$$

Corollaries 1.7–1.8 follow from Theorems 1.1, 1.4 and 1.3 because of the inequality in (1.44). Indeed, apart from the fact that the principal Dirichlet eigenvalue $\lambda(E)$ is decreasing in E rather than increasing, the proofs are identical and we will prove only Corollary 1.8.

Since $\lambda \mapsto I_d^{\text{Dirichlet}}(\lambda)$ is continuous and decreasing on $(0, \lambda_d]$, it suffices to prove (1.31) and to show that $\mathbb{P}(\varphi_d(t)^2 \lambda(t, \rho(t)) \leq \lambda) = t^{-I_d^{\text{Dirichlet}}(\lambda)+o(1)}$ for $\lambda < \lambda_d$.

For (1.31), note that $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$ cannot contain a ball of capacity strictly larger than $\kappa_d(\lambda_d/\lambda(t,\rho(t)))^{(d-2)/2}$: by (1.8) and (5.1), the component of $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$ containing such a ball would have an eigenvalue strictly smaller than $\lambda(t,\rho(t))$. In particular, if $\lambda > \lambda_d$ and $\lambda(t,\rho(t)) \geq \lambda\varphi_d(t)^{-2}$, then $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$ cannot contain a ball of capacity $\kappa_d\, \varphi_d(t)^{d-2}((\lambda_d/\lambda)^{(d-2)/2} + \delta)$ for any $\delta > 0$. Taking δ small enough so that $(\lambda_d/\lambda)^{(d-2)/2}$

$+\delta < 1$, applying Theorem 1.3 with E the ball of capacity $\kappa_d((\lambda_d/\lambda)^{(d-2)/2} + \delta)$, and letting $\delta \downarrow 0$, we obtain (1.31).

Now take $\lambda < \lambda_d$. By Proposition 1.11, apart from an event of negligible probability, every component C of $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$ can be isometrically identified (under its intrinsic metric) with a bounded open subset E of $\mathbb{R}^d$, via $C = x + E$ for some $x \in \mathbb{T}^d$. In particular, $\lambda(C) = \lambda(E)$, and we can apply (1.44) to conclude that $\kappa^*(t, \rho(t)) \geq \operatorname{Cap} E \geq \kappa_d\,(\lambda_d/\lambda(C))^{(d-2)/2}$. Applying Theorem 1.1,

$$\begin{aligned}\mathbb{P}(\varphi_d(t)^2\lambda(t,\rho(t)) \leq \lambda) &\leq \mathbb{P}(\kappa^*(t,\rho(t)) \geq \kappa_d(\lambda_d/\lambda)^{(d-2)/2}\varphi_d(t)^{d-2}) \\ &\leq t^{-I_d^{\text{Dirichlet}}(\lambda)+o(1)}.\end{aligned} \tag{5.2}$$

For the reverse inequality, note that Theorem 1.4 implies that $\mathbb{T}^d \setminus W_{\rho(t)}[0,t]$ contains a ball of capacity $\kappa_d\,\varphi_d(t)^{d-2}(\lambda_d/\lambda)^{(d-2)/2}$ with probability at least $t^{-I_d^{\text{Dirichlet}}(\lambda)-o(1)}$. $\square$

5.3 Proof of Corollary 1.9

Proof. Since $r \mapsto I_d^{\text{inradius}}(r)$ is continuous and strictly increasing on $[1,\infty)$ and is infinite elsewhere, it suffices to verify (1.36) and show $\mathbb{P}(\rho_{\text{in}}(t) > r\varphi_d(t)) = t^{-I_d^{\text{inradius}}(r)+o(1)}$ for $r \geq 1$. But the events $\{\rho_{\text{in}}(t) \leq r\varphi_d(t)\}$ and $\{\rho_{\text{in}}(t) > r\varphi_d(t)\}$ are precisely the event

$$\left\{(x + \varphi_d(t)B(0,r)) \cap W[0,t] \neq \varnothing \ \forall x \in \mathbb{T}^d\right\} \tag{5.3}$$

and its complement

$$\left\{\exists x \in \mathbb{T}^d\colon\ (x + \varphi_d(t)B(0,r)) \cap W[0,t] = \varnothing\right\} \tag{5.4}$$

from Corollary 1.5 and equation (1.26) from Corollary 1.6, with $E = B(0,r)$. $\square$

5.4 Proof of Corollary 1.10

Proof. Recall that $\{\rho_{\text{in}}(t) > \epsilon\} = \{\mathcal{C}_\epsilon > t\}$, so that setting $t = u\psi_d(\epsilon)$, $r = \epsilon/\varphi_d(u\psi_d(\epsilon))$ rewrites the event $\{\mathcal{C}_\epsilon > u\psi_d(\epsilon)\}$ as $\{\rho_{\text{in}}(t) > r\varphi_d(t)\}$. By (1.39), $r \to (u/d)^{1/(d-2)}$ as $\epsilon \downarrow 0$. It follows that $\mathbb{P}(\mathcal{C}_\epsilon > u\psi_d(\epsilon)) = t^{-I_d^{\text{inradius}}((u/d)^{1/(d-2)})+o(1)}$ for $u > d$, since $r \mapsto I_d^{\text{inradius}}(r)$ is continuous on $(1,\infty)$. Noting that $t = \epsilon^{-(d-2)+o(1)}$, this last expression is $\epsilon^{I_d^{\text{cover}}(u)+o(1)}$. A similar argument proves (1.38). Because $u \mapsto I_d^{\text{cover}}(u)$ is continuous and strictly increasing on $[d,\infty)$ and $I_d^{\text{cover}}(v) = \infty$ otherwise, these two facts complete the proof. $\square$

A Hitting probabilities for excursions

A.1 Unconditioned excursions: proof of Lemma 2.5

Proof. Since $R < \frac{1}{2}$, we may consider $x, \xi', \xi, W(t)$ to have values in $\mathbb{R}^d$ instead of $\mathbb{T}^d$. Furthermore, w.l.o.g. we may assume that $x = 0$.

We first remove the effect of conditioning on the exit point $\xi \in \partial B(0,R)$. Define $T = \sup\{t < \zeta\colon\ d(0, W(t)) \leq r\}$ to be the last exit time from $B(0,r)$ before time ζ; note that

$E \cap W[0,\zeta_R] = E \cap W[0,T]$. Let $\tilde{r} \in (r,R)$ and define $\tilde{\tau} = \inf\{t > T\colon\ d(0,W(t)) = \tilde{r}\}$ to be the first hitting time of $\partial B(0,\tilde{r})$ after time T.

Under $\mathbb{P}_{\xi'}$ (i.e., without conditioning on the exit point $W(\zeta_R)$) we can express $(W(t))_{0\leq t\leq \zeta_R}$ as the initial segment $(W(t))_{0\leq t\leq \tilde{\tau}}$ followed by a Brownian motion, conditionally independent given $W(\tilde{\tau})$, started at $\tilde{\xi} = W(\tilde{\tau})$ and conditioned to exit $B(0,R)$ before hitting $B(0,r)$. Let $f_{\tilde{r},R}(\tilde{\xi},\cdot)$ denote the density, with respect to the uniform measure σ_R on $\partial B(0,R)$, of the first hitting point $W(\zeta_R)$ on $\partial B(0,R)$ for this conditioned Brownian motion. Then for $S \subset \partial B(0,R)$ measurable, we have

$$\mathbb{P}_{\xi'}(E \cap W[0,\zeta_R] \neq \varnothing, W(\zeta_R) \in S) = \mathbb{E}_{\xi'}\left(\mathbb{1}_{\{E\cap W[0,T]\neq\varnothing\}} \int_S f_{\tilde{r},R}(W(\tilde{\tau}),\xi)d\sigma_R(\xi)\right). \tag{A.1}$$

From (A.1) it follows that the conditioned measure $\mathbb{P}_{\xi',\xi}$ satisfies

$$\mathbb{P}_{\xi',\xi}(E \cap W[0,\zeta_R] \neq \varnothing) = \frac{\mathbb{E}_{\xi'}\big(\mathbb{1}_{\{E\cap W[0,T]\neq\varnothing\}} f_{\tilde{r},R}(W(\tilde{\tau}),\xi)\big)}{\mathbb{E}_{\xi'}(f_{\tilde{r},R}(W(\tilde{\tau}),\xi))}. \tag{A.2}$$

(More precisely, we would conclude (A.2) for σ_R-a.e. ξ, but by a continuity argument we can take (A.2) to hold for all ξ.)

Now choose $\tilde{r}$ in such a way that $R/\tilde{r} \to \infty$, $\tilde{r}/r \to \infty$, for instance, $\tilde{r} = \sqrt{rR}$. Since $R/\tilde{r} \to \infty$, we have $f_{\tilde{r},R}(\tilde{\xi},\xi) = 1 + o(1)$, uniformly in $\tilde{\xi},\xi$. Therefore

$$\begin{aligned}\mathbb{P}_{\xi',\xi}&(E \cap W[0,\zeta_R] \neq \varnothing) = [1+o(1)]\,\mathbb{P}_{\xi'}(E \cap W[0,\zeta_R] \neq \varnothing)\\ &= [1+o(1)]\,\big(\mathbb{P}_{\xi'}(E \cap W\,[0,\infty) \neq \varnothing) - \mathbb{P}_{\xi'}(E \cap W\,[\zeta_R,\infty) \neq \varnothing)\big).\end{aligned} \tag{A.3}$$

By the Markov property, the last term in (A.3) is the probability of hitting E when starting from some point $W(\zeta_R) \in \partial B(0,R)$ (averaged over the value of $W(\zeta_R)$). Since $R/r \to \infty$, this will be shown to be an error term, and the proof will have been completed once we show that

$$\mathbb{P}_{\xi'}(W\,[0,\infty) \cap E \neq \varnothing) = \frac{\operatorname{Cap} E}{\kappa_d\, r^{d-2}}\,[1+o(1)] \qquad \text{as } r/\epsilon \to \infty. \tag{A.4}$$

Note that (A.4) is essentially the limit in (1.10), taken uniformly over the choice of $E \subset B(0,\epsilon)$.

To show (A.4), let $g_\epsilon(\xi',\cdot)$ denote the density, with respect to the uniform measure σ_ϵ on $\partial B(0,\epsilon)$, of the first hitting point of $\partial B(0,\epsilon)$ for a Brownian motion started at ξ' and conditioned to hit $B(0,\epsilon)$. Then

$$\mathbb{P}_{\xi'}(W\,[0,\infty) \cap E \neq \varnothing) = \frac{\epsilon^{d-2}}{r^{d-2}} \int_{\partial B(0,\epsilon)} \mathbb{P}_y(W\,[0,\infty) \cap E \neq \varnothing)\, g_\epsilon(\xi',y)d\sigma_\epsilon(y). \tag{A.5}$$

Since $r/\epsilon \to \infty$, we have $g_\epsilon(\xi',y) \to 1$ uniformly in ξ', y. Hence (A.4) follows from (A.5) and (2.15). $\square$

A.2 Excursions avoiding an obstacle: proof of Lemma 4.4

Proof. It suffices to bound from below

$$\mathbb{P}_{\xi',\xi}\big(W[0,\zeta_R] \text{ intersects } x + B(\varphi y, \tfrac{1}{2}\rho) \text{ but not } x + (\varphi E)_\rho\big)\,, \tag{A.6}$$

since conditioning on $(x + (\varphi E)_\rho) \cap W[0,\zeta_R] = \varnothing$ can only increase the probability in (A.6). Moreover, as in the proof of Lemma 2.5, we may replace $\mathbb{P}_{\xi',\xi}$ by $\mathbb{P}_{\xi'}$, using now that the densities $f_{\tilde{r},R}$ and g_ϵ are bounded away from 0 and ∞ when $r \leq \frac{1}{2}R$.

Fix $E \in \mathcal{E}_c^\square$, so that $E = \frac{1}{n}A$ for some $A \in \mathcal{A}^\square \cap \mathcal{E}_c$ and $n \in \mathbb{N}$, and fix $\delta > 0$ (we may assume that $\delta < 1/(2n)$). By assumption, E is bounded, say $E \subset B(0,a)$. By adjusting ϵ, we may assume that $\rho/\varphi < a$ (so that $(\varphi E)_\rho \subset B(0, 2a\varphi)$) and $r > 4a\varphi$. We distinguish between three cases:

$y \in B(0,3a) \setminus E_\delta$. Consider $w \in \mathbb{Z}^d \setminus A$. Because $A \in \mathcal{E}_c$, there is a finite path of open cubes with centres $w_0, w_1, \ldots, w_k \in \mathbb{Z}^d$ such that $w_0 \in \mathbb{Z}^d \setminus B(0, 3an)$, $w_k = w$, $d(w_{j-1}, w_j) = 1$ and $\mathrm{int}\big(\cup_{j=0}^k (w_j + [-\frac{1}{2}, \frac{1}{2}]^d)\big) \cap A = \varnothing$. By compactness, the length k of such paths may be taken to be uniformly bounded. Hence, if $\rho/\varphi < \delta/2$, then, given $\xi'' \in \partial B(x, 3a\varphi)$, there is a path $\Gamma \subset B(x, 3a\varphi)$ from ξ'' to $x + \varphi y$ consisting of a finite number of line segments, each of length at most φ, such that $\Gamma_{\delta\varphi/2} \cap (x + (\varphi E)_\rho) = \Gamma_{\delta\varphi/2} \cap (x + \varphi(E_{\rho/\varphi})) = \varnothing$. Moreover, the number of line segments can be taken to be bounded uniformly in y and ξ''. In fact, Γ can be chosen as the union of line segments between points $x + \varphi w_0/n, \ldots, x + \varphi w_k/n$ as above, together with a bounded number of line segments to join ξ'' to $x + \varphi w_0/n$ in $B(x, 3a\varphi) \setminus B(x, 2a\varphi)$ and to join $x + \varphi w_k/n$ to $x + \varphi y$ in the cube $x + (\varphi/n)(w + [-\frac{1}{2}, \frac{1}{2}]^d)$ containing y (see Figure 9)

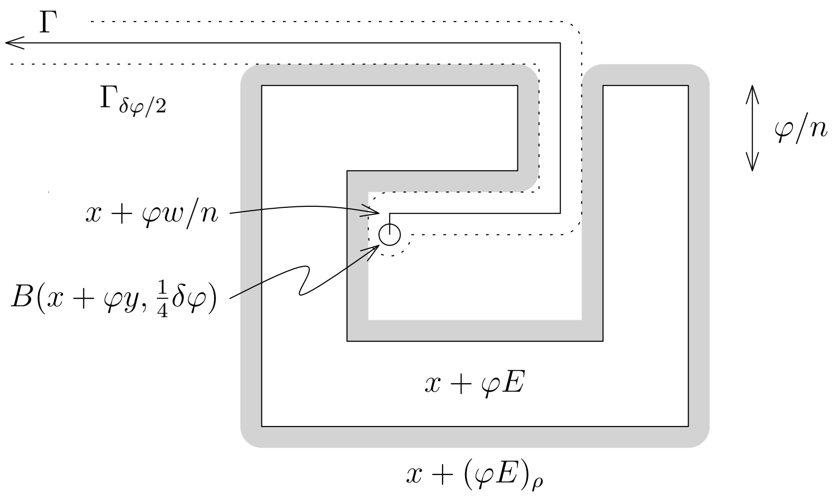

Figure 9: The path Γ reaching $x + \varphi y$. The sets $\Gamma_{\delta\varphi/2}$ and $x + (\varphi E)_\rho = x + \varphi(E_{\rho/\varphi})$ are depicted for the worst-case scenario where the parameters $\rho/\varphi < \delta/2 < 1/4n$ are equal.

From $\xi' \in \partial B(x, r)$, the Brownian path reaches $\partial B(x, 3a\varphi)$ before $\partial B(x, R)$ with probability $(r^{-(d-2)} - R^{-(d-2)})/((3a\varphi)^{-(d-2)} - R^{-(d-2)})$. By our assumptions, this is at least $c_1(\varphi/r)^{d-2}$ for some $c_1 > 0$. Uniformly in the first hitting point ξ'' of $\partial B(x, 3a\varphi)$,

there is a positive probability of hitting $\partial B(x+\varphi y, \frac{1}{4}\delta\varphi)$ via $\Gamma_{\delta\varphi/2}$ before hitting $\partial B(x, 4a\varphi)$. The probability of next hitting $\partial B(x+\varphi y, \frac{1}{2}\rho)$ before $\partial B(x+\varphi y, \frac{1}{2}\delta\varphi)$ is

$$[(\tfrac{1}{4}\delta\varphi)^{-(d-2)} - (\tfrac{1}{2}\delta\varphi)^{-(d-2)}]/[(\tfrac{1}{2}\rho)^{-(d-2)} - (\tfrac{1}{2}\delta\varphi)^{-(d-2)}], \tag{A.7}$$

which is at least $c_2(\rho/\varphi)^{d-2}$ for some $c_2 > 0$. Thereafter there is a positive probability of returning to $\partial B(x, r)$ without hitting $x+(\varphi E)_\rho$, via $\Gamma_{\delta\varphi/2}$. Combining these probabilities gives the required bound.

$\underline{y \in E_\delta \setminus E_{\rho/\varphi}}$. We have $y \in \frac{1}{n}(w + [-\frac{1}{2}, \frac{1}{2}]^d)$ for some $w \in \mathbb{Z}^d$. Write $C_\theta(y, \frac{1}{n}w)$ for the cone with vertex y, central angle θ, and axis the ray from y to $\frac{1}{n}w$. We can choose the angle θ and a constant $c_3 > 0$ small enough (in a manner depending only on d) so that $C_\theta(y, \frac{1}{n}w) \cap E_{\rho/\varphi} \cap B(y, (1+c_3)d(y, w)) = \varnothing$. With θ and c_3 fixed, we can choose $c_4 > 0$ so that every point of $B(\frac{1}{n}w, c_4)$ is a distance at least $c_5 > 0$ from $\partial C_\theta(y, \frac{1}{n}w)$ and $\partial B(y, (1+c_3)d(y, \frac{1}{n}w))$ (see Figure 10).

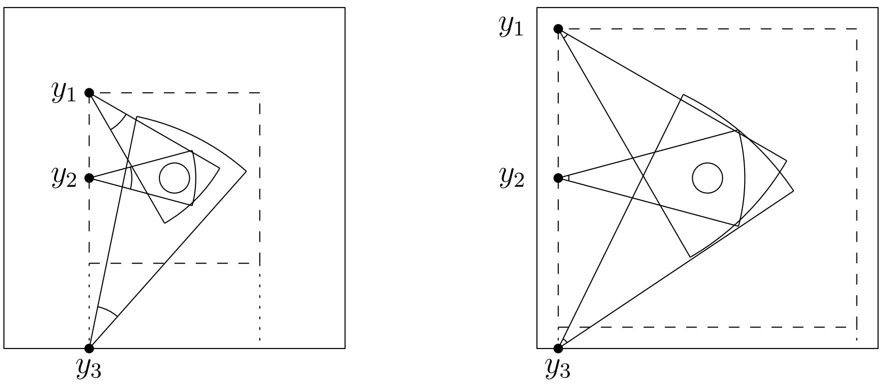

Figure 10: Cones $C_\theta(y, \frac{1}{n}w)$ and parts of balls $B(y, \rho/(2\varphi))$ and $B(y, (1+c_3)d(y, \frac{1}{n}w))$ for three choices of y. The outer square is the cube $\frac{1}{n}(w + [-\frac{1}{2}, \frac{1}{2}]^d)$ containing y. The dashed line shows the greatest possible extent of $E_{\rho/\varphi}$. At least one face of the cube is not contained in E, resulting in a conduit to the outside of the cube (dotted lines). The ball $B(\frac{1}{n}w, c_4)$ in the centre is uniformly bounded away from the sides of the cones and from the other balls. On the left the parameters $\rho/\varphi < 1/4n$ are depicted as equal. On the right is the more relevant situation $\rho/\varphi \ll 1/(4n)$.

Under these conditions, there is a probability at least $c_6(\rho/\varphi)^\alpha$ for a Brownian path started from a point of $\partial B(x+\varphi w/n, c_4\varphi)$ to reach $\partial B(x+\varphi y, \frac{1}{2}\rho)$ before hitting $\partial B(x+\varphi y, \varphi(1+c_3)d(y, w)) \cup \partial(x+\varphi C_\theta(y, w))$, and then to reach $\partial B(x+\varphi y, \varphi d(y, w))$ before hitting $\partial(x+\varphi C_\theta(y, w))$.[5] The rest of the proof proceeds as in the previous case.

$\underline{y \in B(0, r/\varphi) \setminus B(0, 3a)}$. Let $b = d(0, y) \in [3a, r/\varphi]$. The probability that a Brownian path started from ξ' first hits $\partial B(x, b\varphi)$ without hitting $\partial B(x, R)$, then hits

[5] This follows from hitting estimates for Brownian motion in a cone. For instance, via the notation of Burkholder [8, pp. 192–193], the harmonic functions on $C(0, z_0)$ given by $u_1(z) = r_0^{p+d-2}(|z|^{-(p+d-2)} - |z|^p)h(\vartheta)$ and $u_2(z) = |z|^p\, h(\vartheta)$ (with ϑ the angle between z and z_0 and the value $p > 0$ chosen so that $u_1(z) = u_2(z) = 0$ on $\partial C(0, z_0)$) are lower bounds for the probabilities, starting from $z \in C(0, z_0)$, of hitting $\partial B(0, r_0)$ before $\partial B(0, 1) \cup \partial C(0, z_0)$ and of hitting $\partial B(0, 1)$ before $\partial C(0, z_0)$, respectively.

$\partial B(x+\varphi y, \frac{1}{12}b\varphi)$ without hitting $\partial B(x, \frac{2}{3}b\varphi)$, then hits $\partial B(x+\varphi y, \frac{1}{2}\rho)$ before hitting $\partial B(x+\varphi y, \frac{1}{6}b\varphi)$, and finally exits $B(x,R)$ without hitting $\partial B(x, \frac{2}{3}b\varphi)$, is at least $[c_7(b\varphi/r)^{d-2}][c_8][c_9(\rho/(b\varphi))^{d-2}][c_{10}]$. Since $x+(\varphi E)_\rho \subset B(x, 2a\varphi) \subset B(x, \frac{2}{3}b\varphi)$, this is the required bound. $\square$

References

[1] Bandle, C.: Isoperimetric Inequalities and Applications. No. 7 in Monographs and Studies in Mathematics. Pitman (1980)

[2] Benjamini, I., Sznitman, A.S.: Giant component and vacant set for random walk on a discrete torus. J. European Math. Soc. **10**(1), 133–172 (2008)

[3] van den Berg, M.: Heat equation on the arithmetic von Koch snowflake. Probab. Theory Rel. Fields **118**, 17–36 (2000)

[4] van den Berg, M., Bolthausen, E.: Area versus capacity and solidification in the crushed ice model. Probab. Theory Rel. Fields **130**, 69–108 (2004)

[5] van den Berg, M., Bolthausen, E., den Hollander, F.: Moderate deviations for the volume of the Wiener sausage. Ann. Math. **153**(2), 355–406 (2001)

[6] van den Berg, M., Bolthausen, E., den Hollander, F.: Heat content, inradius and spectrum for regions with a Brownian boundary (2012). In preparation

[7] van den Berg, M., den Hollander, F.: Asymptotics for the heat content of a planar region with a fractal polygonal boundary. Proc. London Math. Soc. **78**(3), 627–661 (1999)

[8] Burkholder, D.L.: Exit times of Brownian motion, harmonic majorization, and Hardy spaces. Adv. Math. **26**(2), 182–205 (1977)

[9] Dembo, A., Peres, Y., Rosen, J.: Brownian motion on compact manifolds: cover time and late points. Electron. J. Probab. **8**(15), 1–14 (2003)

[10] Dembo, A., Peres, Y., Rosen, J., Zeitouni, O.: Cover times for Brownian motion and random walks in two dimensions. Ann. Math. **160**(2), 433–464 (2004)

[11] Doob, J.L.: Classical Potential Theory and Its Probabilistic Counterpart. No. 262 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag (1984)

[12] Klarner, D.A.: Cell growth problems. Can. J. Math. **19**, 851–863 (1967)

[13] Levine, L., Peres, Y.: Strong spherical asymptotics for rotor-router aggregation and the divisible sandpile. Potential Anal. **30**, 1–27 (2009)

[14] Matheron, G.: Random Sets and Integral Geometry. Wiley series in probability and mathematical statistics. Wiley (1975)

[15] Mejía Miranda, Y., Slade, G.: The growth constants of lattice trees and lattice animals in high dimensions. Electron. Comm. Probab. **16**(13), 129–136 (2011)

[16] Molchanov, I.: Theory of Random Sets. Probability and its Applications. Springer (2005)

[17] Pólya, G., Szegö, G.: Isoperimetric Inequalities in Mathematical Physics. No. 27 in Annals of Mathematics Studies. Princeton University Press (1951)

[18] Port, S.C., Stone, C.J.: Brownian Motion and Classical Potential Theory. Probability and Mathematical Statistics. Academic Press (1978)

[19] Sidoravicius, V., Sznitman, A.S.: Percolation for the vacant set of random interlacements. Comm. Pure Appl. Math. **62**(6), 831–858 (2009)

[20] Sznitman, A.: Brownian Motion, Obstacles and Random Media. Springer Monographs in Mathematics. Springer (1998)

[21] Sznitman, A.S.: Vacant set of random interlacements and percolation. Ann. Math. **171**(3), 2039–2087 (2010)

[22] Teixeira, A., Windisch, D.: On the fragmentation of a torus by random walk. Comm. Pure Appl. Math. **64**(12), 1599–1646 (2011)